HAAR EXPECTATIONS OF RATIOS OF RANDOM CHARACTERISTIC POLYNOMIALS

by

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Abstract. — We compute Haar ensemble averages of ratios of random characteristic polynomials for the classical Lie groups $K = \mathrm{O}_N$, SO_N, and USp_N. To that end, we start from the Clifford-Weyl algebra in its canonical realization on the complex $\mathscr{A}_V$ of holomorphic differential forms for a $\mathbb{C}$-vector space V_0. From it we construct the Fock representation of an orthosymplectic Lie superalgebra $\mathfrak{osp}$ associated to V_0. Particular attention is paid to defining Howe's oscillator semigroup and the representation that partially exponentiates the Lie algebra representation of $\mathfrak{sp} \subset \mathfrak{osp}$. In the process, by pushing the semigroup representation to its boundary and arguing by continuity, we provide a construction of the Shale-Weil-Segal representation of the metaplectic group.

To deal with a product of n ratios of characteristic polynomials, we let $V_0 = \mathbb{C}^n \otimes \mathbb{C}^N$ where $\mathbb{C}^N$ is equipped with the standard K-representation, and focus on the subspace $\mathscr{A}_V^K$ of K-equivariant forms. By Howe duality, this is a highest-weight irreducible representation of the centralizer $\mathfrak{g}$ of $\mathrm{Lie}(K)$ in $\mathfrak{osp}$. We identify the K-Haar expectation of n ratios with the character of this $\mathfrak{g}$-representation, which we show to be uniquely determined by analyticity, Weyl group invariance, certain weight constraints and a system of differential equations coming from the Laplace-Casimir invariants of $\mathfrak{g}$. We find an explicit solution to the problem posed by all these conditions. In this way, we prove that the said Haar expectations are expressed by a Weyl-type character formula for *all* integers $N \geq 1$. This completes earlier work of Conrey, Farmer, and Zirnbauer for the case of U_N.

Contents

1. Introduction

In this article we derive an explicit formula for the average

$$I(t) := \int_K Z(t,k)\,dk$$

where K is one of the classical compact Lie groups O_N, SO_N, or USp_N equipped with Haar measure dk of unit mass $\int_K dk = 1$ and

$$Z(t,k) := \prod_{j=1}^{n} \frac{\mathrm{Det}(\mathrm{e}^{\frac{\mathrm{i}}{2}\psi_j}\mathrm{Id}_N - \mathrm{e}^{-\frac{\mathrm{i}}{2}\psi_j}\,k)}{\mathrm{Det}(\mathrm{e}^{\frac{1}{2}\phi_j}\mathrm{Id}_N - \mathrm{e}^{-\frac{1}{2}\phi_j}\,k)}$$

depends on a set of complex parameters $t := (\mathrm{e}^{\mathrm{i}\psi_1},\ldots,\mathrm{e}^{\mathrm{i}\psi_n},\mathrm{e}^{\phi_1},\ldots,\mathrm{e}^{\phi_n})$, which satisfy $\mathfrak{Re}\,\phi_j > 0$ for all $j = 1,\ldots,n$. The case of $K = \mathrm{U}_N$ is handled in [**4**]. Note that

$$Z(t,k) = \mathrm{e}^{\lambda_N} \prod_{j=1}^{n} \frac{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\mathrm{i}\psi_j}\,k)}{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\phi_j}\,k)}$$

with $\lambda_N = \frac{N}{2}\sum_{j=1}^{n}(\mathrm{i}\psi_j - \phi_j)$. This means that $Z(t,k)$ is a product of ratios of characteristic polynomials, which explains the title of the article.

The Haar average $I(t)$ can be regarded as the (numerical part of the) character of an irreducible representation of a Lie supergroup $(\mathfrak{g},G)$ restricted to a suitable subset of a maximal torus of G. The Lie superalgebra $\mathfrak{g}$ is the Howe dual partner of the compact group K in an orthosymplectic Lie superalgebra $\mathfrak{osp}$. It is naturally represented on a certain infinite-dimensional spinor-oscillator module $\mathfrak{a}(V)$ – more concretely, the complex of holomorphic differential forms on the vector space $\mathbb{C}^n \otimes \mathbb{C}^N$ – and the irreducible representation is that on the subspace $\mathfrak{a}(V)^K$ of K-equivariant forms.

To even define the character, we must exponentiate the representation of the Lie algebra part of $\mathfrak{osp}$ on $\mathfrak{a}(V)$. This requires going to a completion $\mathscr{A}_V$ of $\mathfrak{a}(V)$, and can only be done partially. Nevertheless, the represented semigroup contains enough structure to derive Laplace-Casimir differential equations for its character.

Our explicit formula for $I(t)$ looks exactly like a classical Weyl formula and is derived in terms of the roots of the Lie superalgebra $\mathfrak{g}$ and the Weyl group W. Let us state this formula for $K = \mathrm{O}_N$, USp_N without going into the details of the λ-positive even and odd roots $\Delta^+_{\lambda,0}$ and $\Delta^+_{\lambda,1}$ and the Weyl group W (see §4.3.2 for precise formulas). If W_λ is the isotropy subgroup of W fixing the highest weight $\lambda = \lambda_N$, then

$$I(t) = \sum_{[w]\in \mathrm{W}/\mathrm{W}_\lambda} \mathrm{e}^{w(\lambda_N)} \frac{\prod_{\beta\in\Delta^+_{\lambda,1}}(1-\mathrm{e}^{-w(\beta)})}{\prod_{\alpha\in\Delta^+_{\lambda,0}}(1-\mathrm{e}^{-w(\alpha)})}(\ln t)\,. \tag{1.1}$$

To prove this formula we establish certain properties of $I(t)$ which uniquely characterize it and are satisfied by the right-hand side. These are a weight expansion of $I(t)$ (see Corollary 4.1), restrictions on the set of weights (see Corollary 2.3), and the fact that $I(t)$ is annihilated by certain invariant differential operators (see Corollary 4.2).

As was stated above, the case $K = \mathrm{U}_N$ is treated in [**4**]. Here, we restrict to the compact groups $K = \mathrm{O}_N$, $K = \mathrm{USp}_N$, and $K = \mathrm{SO}_N$. The cases $K = \mathrm{O}_N$ and $K =$

USp_N can be treated simultaneously. Having established formula (1.1) for $K = \mathrm{O}_N$, the following argument gives a similar result for $K = \mathrm{SO}_N$. Let dk_{O} and dk_{SO} be the unit mass Haar measures for O_N and SO_N, respectively. The O_N-measure $(1 + \mathrm{Det}\, k)\, dk_{\mathrm{O}}$ has unit mass on SO_N and zero mass on $\mathrm{O}_N^- = \mathrm{O}_N \setminus \mathrm{SO}_N$. It is SO_N-invariant. Now,

$$\begin{aligned} I_{\mathrm{SO}_N}(t) &= \int_{\mathrm{SO}_N} Z(t,k)\, dk_{\mathrm{SO}} = \int_{\mathrm{O}_N} Z(t,k)(1 + \mathrm{Det}\, k)\, dk_{\mathrm{O}} \\ &= \int_{\mathrm{O}_N} Z(t,k)\, dk_{\mathrm{O}} + \int_{\mathrm{O}_N} Z(t,k)\mathrm{Det}(k)\, dk_{\mathrm{O}} = I_{\mathrm{O}_N}(t) + (-1)^N I_{\mathrm{O}_N}(t') \end{aligned}$$

with $t' = (\mathrm{e}^{-\mathrm{i}\psi_1}, \mathrm{e}^{\mathrm{i}\psi_2}, \ldots, \mathrm{e}^{\mathrm{i}\psi_n}, \mathrm{e}^{\phi_1}, \ldots, \mathrm{e}^{\phi_n})$, since $\mathrm{Det}(k)\, Z(t,k) = (-1)^N Z(t',k)$.

1.1. Comparison with results of other approaches. — To facilitate the comparison with related work, we now present our final results in the following explicit form. Let $x_j := \mathrm{e}^{-\mathrm{i}\psi_j}$ and $y_l := \mathrm{e}^{-\phi_l}$. Consider first the case of the unitary symplectic group $K = \mathrm{USp}_N$ (where $N \in 2\mathbb{N}$). Then for any pair of non-negative integers p, q in the range $q - p \le N + 1$ one directly infers from (1.1) the formula

$$\int_{\mathrm{USp}_N} \frac{\prod_{k=1}^p \mathrm{Det}(1 - x_k u)}{\prod_{l=1}^q \mathrm{Det}(1 - y_l u)}\, du = \sum_{\varepsilon \in \{\pm 1\}^p} \frac{\prod_{k=1}^p x_k^{\frac{N}{2}(1-\varepsilon_k)} \prod_{l=1}^q (1 - x_k^{\varepsilon_k} y_l)}{\prod_{k \le k'} (1 - x_k^{\varepsilon_k} x_{k'}^{\varepsilon_{k'}}) \prod_{l < l'} (1 - y_l y_{l'})}\,.$$

The sum on the right-hand side is over sign configurations $\varepsilon \equiv (\varepsilon_1, \ldots, \varepsilon_p) \in \{\pm 1\}^p$. The proof proceeds by induction in p, starting from the result (1.1) for $p = q$ and sending $x_p \to 0$ to pass from p to $p-1$. In published recent work [**5, 3**] the same formula was derived under the more restrictive condition $q \le N/2$. In [**3**] this unwanted restriction on the parameter range came about because the numerator and denominator on the left-hand side were expanded *separately*, ignoring the supersymmetric Howe duality (see §2 of the present paper) of the problem at hand.

For $K = \mathrm{SO}_N$ the same induction process starting from (1.1) yields the result

$$\int_{\mathrm{SO}_N} \frac{\prod_{k=1}^p \mathrm{Det}(1 - x_k u)}{\prod_{l=1}^q \mathrm{Det}(1 - y_l u)}\, du = \sum_{\varepsilon \in \{\pm 1\}^p} \frac{\prod_{k=1}^p (\varepsilon_k x_k)^{\frac{N}{2}(1-\varepsilon_k)} \prod_{l=1}^q (1 - x_k^{\varepsilon_k} y_l)}{\prod_{k < k'} (1 - x_k^{\varepsilon_k} x_{k'}^{\varepsilon_{k'}}) \prod_{l \le l'} (1 - y_l y_{l'})}$$

as long as $q - p \le N - 1$. Please note that this includes even the case of the trivial group $K = \mathrm{SO}_1 = \{\mathrm{Id}\}$ with any $p = q > 0$. For $K = \mathrm{O}_N$ one has an analogous result where the sum on the right-hand side is over ε with an *even* number of sign reversals.

The very same formulas for SO_N and O_N were derived in the recent literature [**5, 3**] but, again, only in the much narrower range $q \le \mathrm{Int}[N/2]$.

1.2. Howe duality and weight expansion. — To find an explicit expression for the integral $I(t)$, we first of all observe that the integrand $Z(t,k)$ is the supertrace of a representation ρ of a semigroup $(T_1 \times T_+) \times K$ on the spinor-oscillator module $\mathfrak{a}(V)$ (cf. Lemma 4.5). More precisely, we start with the standard K-representation space $\mathbb{C}^N$, the $\mathbb{Z}_2$-graded vector space $U = U_0 \oplus U_1$ with $U_s \simeq \mathbb{C}^n$, and the abelian semigroup

$$T_1 \times T_+ := \{(\mathrm{diag}(\mathrm{e}^{\mathrm{i}\psi_1}, \ldots, \mathrm{e}^{\mathrm{i}\psi_n}), \mathrm{diag}(\mathrm{e}^{\phi_1}, \ldots, \mathrm{e}^{\phi_n})) \mid \mathfrak{Re}\, \phi_j > 0\,,\, j = 1, \ldots, n\}$$

of diagonal transformations in $\mathrm{GL}(U_1) \times \mathrm{GL}(U_0)$. We then consider the vector space $V := U \otimes \mathbb{C}^N$ which is $\mathbb{Z}_2$-graded by $V_s = U_s \otimes \mathbb{C}^N$, the infinite-dimensional spinor-oscillator module $\mathfrak{a}(V) := \wedge(V_1^*) \otimes \mathrm{S}(V_0^*)$, and a representation ρ of $(T_1 \times T_+) \times K$ on $\mathfrak{a}(V)$. We also let $V \oplus V^* =: W = W_0 \oplus W_1$ (not the Weyl group).

Averaging the product of ratios $Z(t,k)$ with respect to the compact group K corresponds to the projection from $\mathfrak{a}(V)$ onto the vector space $\mathfrak{a}(V)^K$ of K-invariants (Corollary 4.1). Now, Howe duality (Proposition 2.2) implies that $\mathfrak{a}(V)^K$ is the representation space for an irreducible highest-weight representation ρ_* of the Howe dual partner $\mathfrak{g}$ of K in the orthosymplectic Lie superalgebra $\mathfrak{osp}(W)$. This representation ρ_* is constructed by realizing $\mathfrak{g} \subset \mathfrak{osp}(W)$ as a subalgebra in the space of degree-two elements of the Clifford-Weyl algebra $\mathfrak{q}(W)$. Precise definitions of these objects, their relationships, and the Howe duality statement can be found in §2.

Using the decomposition $\mathfrak{a}(V)^K = \oplus_{\gamma \in \Gamma} V_\gamma$ into weight spaces, Howe duality leads to the weight expansion $I(\mathrm{e}^H) = \mathrm{STr}_{\mathfrak{a}(V)^K}\, \mathrm{e}^{\rho_*(H)} = \sum_{\gamma \in \Gamma} B_\gamma \mathrm{e}^{\gamma(H)}$ for $t = \mathrm{e}^H \in T_1 \times T_+$. Here STr denotes the supertrace. There are strong restrictions on the set of weights Γ. Namely, if $\gamma \in \Gamma$, then $\gamma = \sum_{j=1}^n (\mathrm{i} m_j \psi_j - n_j \phi_j)$ and $-\frac{N}{2} \le m_j \le \frac{N}{2} \le n_j$ for all j. The coefficients $B_\gamma = \mathrm{STr}_{V_\gamma}(\mathrm{Id})$ are the dimensions of the weight spaces (multiplied by parity). Note that the set of weights of the representation ρ_* of $\mathfrak{g}$ on $\mathfrak{a}(V)^K$ is infinite.

1.3. Group representation and differential equations. — Before outlining the strategy for computing our character in the infinite-dimensional setting of representations of Lie superalgebras and groups, we recall the classical situation where ρ_* is an irreducible finite-dimensional representation of a reductive Lie algebra $\mathfrak{g}$ and ρ is the corresponding Lie group representation of the complex reductive group G. In that case the character χ of ρ, which automatically exists, is the trace $\mathrm{Tr}\,\rho$, which is a radial eigenfunction of every Laplace-Casimir operator. These differential equations can be completely understood by their behavior on a maximal torus of G.

In our case we must consider the infinite-dimensional irreducible representation ρ_* of the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}$ on $\mathfrak{a}(V)^K$. Casimir elements, Laplace-Casimir operators of $\mathfrak{osp}$, and their radial parts have been described by Berezin [**1**]. In the situation $U_0 \simeq U_1$ at hand we have the additional feature that every $\mathfrak{osp}$-Casimir element I can be expressed as a bracket $I = [\partial, F]$ where ∂ is the holomorphic exterior derivative when we view $\mathfrak{a}(V)^K$ as the complex of K-equivariant holomorphic differential forms on V_0.

To benefit from Berezin's theory of radial parts, we construct a radial superfunction χ which is defined on an open set containing the torus $T_1 \times T_+$ such that its numerical part satisfies $\mathrm{num}\chi(t) = I(t)$ for all $t \in T_1 \times T_+$. If we had a representation (ρ_*, ρ) of a Lie supergroup $(\mathfrak{osp}, G)$ at our disposal, we could define χ to be its character, i.e.

$$\chi(g) \stackrel{?}{=} \mathrm{STr}_{\mathfrak{a}(V)^K}\, \rho(g)\, \mathrm{e}^{\sum_j \xi_j \rho_*(\Xi_j)}$$

(see §4.1.4). Since we don't have such a representation, our idea is to define χ as a character on a totally real submanifold M of maximal dimension which contains a real form of $T_1 \times T_+$ and is invariant with respect to conjugation by a real form $G_{\mathbb{R}}$ of G, and then to extend χ by analytic continuation.

Thinking classically we consider the even part of the Lie superalgebra $\mathfrak{osp}(W_0 \oplus W_1)$, which is the Lie algebra $\mathfrak{o}(W_1) \oplus \mathfrak{sp}(W_0)$. The real structures at the Lie supergroup level come from a real form $W_\mathbb{R}$ of W. The associated real forms of $\mathfrak{o}(W_1)$ and $\mathfrak{sp}(W_0)$ are the real orthogonal Lie algebra $\mathfrak{o}(W_{1,\mathbb{R}})$ and the real symplectic Lie algebra $\mathfrak{sp}(W_{0,\mathbb{R}})$. These are defined in such a way that the elements in $\mathfrak{o}(W_{1,\mathbb{R}}) \oplus \mathfrak{sp}(W_{0,\mathbb{R}})$ and $\mathrm{i}W_\mathbb{R}$ are mapped as elements of the Clifford-Weyl algebra via the spinor-oscillator representation to anti-Hermitian operators on $\mathfrak{a}(V)$ with respect to a compatibly defined unitary structure. In this context we frequently use the unitary representation of the real Heisenberg group $\exp(\mathrm{i}W_{0,\mathbb{R}}) \times \mathrm{U}_1$ on the completion $\mathscr{A}_V$ of the module $\mathfrak{a}(V)$.

Since $\wedge(V_1^*)$ has finite dimension, exponentiating the spinor representation of $\mathfrak{o}(W_{1,\mathbb{R}})$ causes no difficulties. This results in the spinor representation of $\mathrm{Spin}(W_{1,\mathbb{R}})$, a 2:1 covering of the compact group $\mathrm{SO}(W_{1,\mathbb{R}})$. So in this case one easily constructs a representation $R_1 : \mathrm{Spin}(W_{1,\mathbb{R}}) \to \mathrm{U}(\mathfrak{a}(V))$ which is compatible with $\rho_*|_{\mathfrak{o}(W_{1,\mathbb{R}})}$.

Exponentiating the oscillator representation of $\mathfrak{sp}(W_{0,\mathbb{R}})$ on the infinite-dimensional vector space $\mathrm{S}(V_0^*)$ requires more effort. In §3.4, following Howe [**8**], we construct the Shale-Weil-Segal representation $R' : \mathrm{Mp}(W_{0,\mathbb{R}}) \to \mathrm{U}(\mathscr{A}_V)$ of the metaplectic group $\mathrm{Mp}(W_{0,\mathbb{R}})$ which is the 2:1 covering group of the real symplectic group $\mathrm{Sp}(W_{0,\mathbb{R}})$. This is compatible with $\rho_*|_{\mathfrak{sp}(W_{0,\mathbb{R}})}$. Altogether we see that the even part of the Lie superalgebra representation integrates to $G_\mathbb{R} = \mathrm{Spin}(W_{1,\mathbb{R}}) \times_{\mathbb{Z}_2} \mathrm{Mp}(W_{0,\mathbb{R}})$.

The construction of R' uses a limiting process coming from the oscillator semigroup $\widetilde{\mathrm{H}}(W_0^s)$, which is the double covering of the contraction semigroup $\mathrm{H}(W_0^s) \subset \mathrm{Sp}(W_0)$ and has $\mathrm{Mp}(W_{0,\mathbb{R}})$ in its boundary. Furthermore, we have $\widetilde{\mathrm{H}}(W_0^s) = \mathrm{Mp}(W_{0,\mathbb{R}}) \times M$ where M is an analytic totally real submanifold of maximal dimension which contains a real form of the torus T_+ (see §3.2). The representation $R_0 : \widetilde{\mathrm{H}}(W_0^s) \to \mathrm{End}(\mathscr{A}_V)$ constructed in §3.3 facilitates the definition of the representation R' and of the character χ in §4.2. It should be underlined that Proposition 3.24 ensures convergence of the superfunction $\chi(h)$, which is defined as a supertrace and exists for all $h \in \widetilde{\mathrm{H}}(W_0^s)$.

On that basis, the key idea of our approach is to exploit the fact that every Casimir invariant $I \in \mathsf{U}(\mathfrak{g})$ is exact in the sense that $I = [\partial, F]$. By a standard argument, this exactness property implies that every such invariant I vanishes in the spinor-oscillator representation. This result in turn implies for our character χ the differential equations $D(I)\chi = 0$ where $D(I)$ is the Laplace-Casimir operator representing I. By drawing on Berezin's theory of radial parts, we derive a system of differential equations which in combination with certain other properties ultimately determines χ.

In the case of $K = \mathrm{O}_N$ the Lie group associated to the even part of the real form of the Howe partner $\mathfrak{g}$ is embedded in a simple way in the full group $G_\mathbb{R}$ described above. It is itself just a lower-dimensional group of the same form. In the case of $K = \mathrm{USp}_N$ a sort of reversing procedure takes places and the analogous real form is $\mathrm{USp}_{2n} \times \mathrm{SO}^*_{2n}$. Nevertheless, the precise data which are used as input into the series developments, the uniqueness theorem and the final calculations of χ are essentially the same in the two cases. Therefore there is no difficulty handling them simultaneously.

2. Howe dual pairs in the orthosymplectic Lie superalgebra

In this chapter we collect some foundational information from representation theory. Basic to our work is the orthosymplectic Lie superalgebra, $\mathfrak{osp}$, in its realization as the space spanned by supersymmetrized terms of degree two in the Clifford-Weyl algebra. Representing the latter by its fundamental representation on the spinor-oscillator module, one gets a representation of $\mathfrak{osp}$ and of all Howe dual pairs inside of $\mathfrak{osp}$. Roots and weights of the relevant representations are described in detail.

2.1. Notion of Lie superalgebra. — A $\mathbb{Z}_2$-*grading* of a vector space V over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a decomposition $V = V_0 \oplus V_1$ of V into the direct sum of two $\mathbb{K}$-vector spaces V_0 and V_1. The elements in $(V_0 \cup V_1) \setminus \{0\}$ are called *homogeneous*. The *parity function* $|\,| : (V_0 \cup V_1) \setminus \{0\} \to \mathbb{Z}_2$, $v \in V_s \mapsto |v| = s$, assigns to a homogeneous element its parity. We write $V \simeq \mathbb{K}^{p|q}$ if $\dim_{\mathbb{K}} V_0 = p$ and $\dim_{\mathbb{K}} V_1 = q$.

A *Lie superalgebra* over $\mathbb{K}$ is a $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a bilinear map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

1. $[\mathfrak{g}_s, \mathfrak{g}_{s'}] \subset \mathfrak{g}_{s+s'}$, i.e., $|[X,Y]| = |X| + |Y|$ (mod 2) for homogeneous elements X, Y.
2. Skew symmetry: $[X,Y] = -(-1)^{|X||Y|}[Y,X]$ for homogeneous X, Y.
3. Jacobi identity, which means that $\mathrm{ad}(X) = [X,\,]$: $\mathfrak{g} \to \mathfrak{g}$ is a (super-)derivation:

$$\mathrm{ad}(X)\,[Y,Z] = [\mathrm{ad}(X)Y, Z] + (-1)^{|X||Y|}[Y, \mathrm{ad}(X)Z]\ .$$

***Example 2.1* ($\mathfrak{gl}(V)$). —** Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space. There is a canonical $\mathbb{Z}_2$-grading $\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$ induced by the grading of V:

$$\mathrm{End}(V)_s := \{X \in \mathrm{End}(V) \mid \forall s' \in \mathbb{Z}_2 : X(V_{s'}) \subset V_{s+s'}\}\ .$$

The bilinear extension of $[X,Y] := XY - (-1)^{|X||Y|}YX$ for homogeneous elements $X, Y \in \mathrm{End}(V)$ to a bilinear map $[\,,\,] : \mathrm{End}(V) \times \mathrm{End}(V) \to \mathrm{End}(V)$ gives $\mathrm{End}(V)$ the structure of a Lie superalgebra, namely $\mathfrak{gl}(V)$. The Jacobi identity in this case is a direct consequence of the associativity $(XY)Z = X(YZ)$ and the definition of $[X,Y]$.

In fact, for every $\mathbb{Z}_2$-graded associative algebra $\mathscr{A}$ the bracket $[\,,\,] : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ defined by $[X,Y] = XY - (-1)^{|X||Y|}YX$ satisfies the Jacobi identity.

***Example 2.2* ($\mathfrak{u}(V)$). —** Given a complex $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$, consider the complex Lie superalgebra $\mathrm{End}(V) = \mathfrak{gl}(V)$. Equip V with a unitary structure so that $V_0 \perp V_1$. For $X \in \mathrm{End}(V)$ we have the unique decomposition $X = \sum_{s=0}^{1} X_s$ with $X_s \in \mathfrak{gl}(V)_s$. Let $\sigma : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ be the complex anti-linear involution defined by $\sigma(X)_s = -\mathrm{i}^s \overline{X}_s^{\mathrm{t}}$, where $X \mapsto \overline{X}^{\mathrm{t}}$ (complex conjugation and transpose) is defined by the unitary structure. σ is an automorphism of the Lie superalgebra $\mathfrak{gl}(V)$. Indeed,

$$\begin{aligned}\sigma([X,Y]) &= \textstyle\sum_{s,s'} \sigma(X_s Y_{s'}) - \sum_{s,s'} (-1)^{ss'} \sigma(Y_{s'} X_s) \\ &= -\textstyle\sum_{s,s'} \mathrm{i}^{(s-s')^2} \left(\overline{Y}_{s'}^{\mathrm{t}} \overline{X}_s^{\mathrm{t}} - (-1)^{ss'} \overline{X}_s^{\mathrm{t}} \overline{Y}_{s'}^{\mathrm{t}} \right)\ ,\end{aligned}$$

which is the same as

$$\begin{aligned} &\textstyle\sum_{s,s'} \mathrm{i}^{s^2+s'^2} \left(\overline{X}_s^{\mathrm{t}} \overline{Y}_{s'}^{\mathrm{t}} - (-1)^{ss'} \overline{Y}_{s'}^{\mathrm{t}} \overline{X}_s^{\mathrm{t}}\right) \\ = \;&\textstyle\sum_{s,s'} \sigma(X)_s \sigma(Y)_{s'} - \sum_{s,s'} (-1)^{ss'} \sigma(Y)_{s'} \sigma(X)_s = [\sigma(X), \sigma(Y)] \,. \end{aligned}$$

The set of fixed points $\mathrm{Fix}(\sigma) = \{X \in \mathfrak{gl}(V) \mid \sigma(X) = X\}$ is a real Lie superalgebra, a real form of $\mathfrak{gl}(V)$ called the unitary Lie superalgebra $\mathfrak{u} := \mathfrak{u}(V)$. Note that $\mathfrak{u}_0$ is the compact real form $\mathfrak{u}(V_0) \oplus \mathfrak{u}(V_1)$ of the complex Lie algebra $\mathfrak{gl}(V_0) \oplus \mathfrak{gl}(V_1)$.

Example 2.3 **($\mathfrak{osp}(V \oplus V^*)$).** — Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space and put $W := V \oplus V^*$. The $\mathbb{Z}_2$-grading of V induces a $\mathbb{Z}_2$-grading $W = W_0 \oplus W_1$ in the obvious manner: $W_0 = V_0 \oplus V_0^*$ and $W_1 = V_1 \oplus V_1^*$. Then consider the canonical alternating bilinear form A on W_0,

$$A : \, W_0 \times W_0 \to \mathbb{K} \,, \quad (v + \varphi \,, v' + \varphi') \mapsto \varphi'(v) - \varphi(v') \,,$$

and the canonical symmetric bilinear form S on W_1,

$$S : \, W_1 \times W_1 \to \mathbb{K} \,, \quad (v + \varphi \,, v' + \varphi') \mapsto \varphi'(v) + \varphi(v') \,.$$

The *orthosymplectic form* of W is the non-degenerate bilinear form $Q : W \times W \to \mathbb{K}$ defined as the orthogonal sum $Q = A + S$:

$$Q(w_0 + w_1, w_0' + w_1') = A(w_0, w_0') + S(w_1, w_1') \quad (w_s \,, w_s' \in W_s) \,.$$

Note the exchange symmetry $Q(w, w') = -(-1)^{|w||w'|} Q(w', w)$ for $w, w' \in W_0 \cup W_1$.

Given Q, define a complex linear bijection $\tau : \mathrm{End}(W) \to \mathrm{End}(W)$ by the equation

$$Q(\tau(X)w, w') + (-1)^{|X||w|} Q(w, Xw') = 0 \tag{2.1}$$

for all $w, w' \in W_0 \cup W_1$. It is easy to check that τ has the property

$$\tau(XY) = -(-1)^{|X||Y|} \tau(Y) \tau(X) \,,$$

which implies that τ is an involutory automorphism of the Lie superalgebra $\mathfrak{gl}(W)$ with bracket $[X, Y] = XY - (-1)^{|X||Y|} YX$. Hence the subspace $\mathfrak{osp}(W) \subset \mathrm{End}(W)$ of τ-fixed points is closed w.r.t. that bracket; it is called the (complex) *orthosymplectic Lie superalgebra* of W.

Example 2.4 **(Jordan-Heisenberg algebra)**. — Using the notation of Example 2.3, consider the vector space $\widetilde{W} := W \oplus \mathbb{K}$ and take it to be $\mathbb{Z}_2$-graded by $\widetilde{W}_0 = W_0 \oplus \mathbb{K}$ and $\widetilde{W}_1 = W_1$. Define a bilinear mapping $[\,,] : \widetilde{W} \times \widetilde{W} \to \widetilde{W}$ by

$$[\mathbb{K}, \widetilde{W}] = [\widetilde{W}, \mathbb{K}] = 0 \,, \quad [W, W] \subset \mathbb{K} \,, \quad [w, w'] = Q(w, w') \quad (w, w' \in W) \,.$$

By the basic properties of the orthosymplectic form Q, the vector space $\widetilde{W}$ equipped with this bracket is a Lie superalgebra – the so-called Jordan-Heisenberg algebra. Note that $\widetilde{W}$ is two-step nilpotent, i.e., $[\widetilde{W}, [\widetilde{W}, \widetilde{W}]] = 0$.

2.1.1. Supertrace. — Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space, and recall the decomposition $\mathrm{End}(V) = \bigoplus_{s,t} \mathrm{Hom}(V_s, V_t)$. For $X \in \mathrm{End}(V)$, we denote by $X = \sum_{s,t} X_{ts}$ the corresponding decomposition of an operator. The *supertrace* on V is the linear function

$$\mathrm{STr}: \ \mathrm{End}(V) \to \mathbb{K}\,, \quad X \mapsto \mathrm{Tr}\, X_{00} - \mathrm{Tr}\, X_{11} = \sum\nolimits_s (-1)^s \mathrm{Tr}\, X_{ss}\,.$$

(If $\dim V = \infty$, then usually the domain of definition of STr must be restricted.)

An ad-*invariant bilinear form* on a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ over $\mathbb{K}$ is a bilinear mapping $B: \ \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ with the properties

1. $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are B-orthogonal to each other ;
2. B is symmetric on $\mathfrak{g}_0$ and skew on $\mathfrak{g}_1$;
3. $B([X,Y],Z) = B(X,[Y,Z])$ for all $X,Y,Z \in \mathfrak{g}$.

We will repeatedly use the following direct consequences of the definition of STr.

Lemma 2.1. — *If $\mathfrak{g}$ is a Lie superalgebra in* $\mathrm{End}(V)$, *the trace form* $B(X,Y) = \mathrm{STr}\,(XY)$ *is an* ad-*invariant bilinear form. One has* $\mathrm{STr}\,[X,Y] = 0$.

Recalling the setting of Example 2.3, note that the supertrace for $W = V \oplus V^*$ is odd under the $\mathfrak{gl}$-automorphism τ fixing $\mathfrak{osp}(W)$, i.e., $\mathrm{STr}_W \circ \tau = -\mathrm{STr}_W$. It follows that $\mathrm{STr}_W X = 0$ for any $X \in \mathfrak{osp}(W)$. Moreover, $\mathrm{STr}_W(X_1 X_2 \cdots X_{2n+1}) = 0$ for any product of an odd number of $\mathfrak{osp}$-elements.

2.1.2. Universal enveloping algebra. — Let $\mathfrak{g}$ be a Lie superalgebra with bracket $[\,,\,]$. The universal enveloping algebra $\mathsf{U}(\mathfrak{g})$ is defined as the quotient of the tensor algebra $\mathsf{T}(\mathfrak{g}) = \oplus_{n=0}^{\infty} \mathsf{T}^n(\mathfrak{g})$ by the two-sided ideal $\mathsf{J}(\mathfrak{g})$ generated by all combinations

$$X \otimes Y - (-1)^{|X||Y|} Y \otimes X - [X,Y]$$

for homogeneous $X, Y \in \mathsf{T}^1(\mathfrak{g}) \equiv \mathfrak{g}$. If $\mathsf{U}_n(\mathfrak{g})$ is the image of $\mathsf{T}_n(\mathfrak{g}) := \oplus_{k=0}^{n} \mathsf{T}^k(\mathfrak{g})$ under the projection $\mathsf{T}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g}) = \mathsf{T}(\mathfrak{g})/\mathsf{J}(\mathfrak{g})$, the algebra $\mathsf{U}(\mathfrak{g})$ is filtered by $\mathsf{U}(\mathfrak{g}) = \cup_{n=0}^{\infty} \mathsf{U}_n(\mathfrak{g})$. The $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ gives rise to a $\mathbb{Z}_2$-grading of $\mathsf{T}(\mathfrak{g})$ by

$$|X_1 \otimes X_2 \otimes \cdots \otimes X_n| = \sum\nolimits_{i=1}^{n} |X_i| \quad \text{(for homogenous } X_i \in \mathfrak{g})\,,$$

and this in turn induces a canonical $\mathbb{Z}_2$-grading of $\mathsf{U}(\mathfrak{g})$.

One might imagine introducing various bracket operations on $\mathsf{T}(\mathfrak{g})$ and/or $\mathsf{U}(\mathfrak{g})$. However, in view of the canonical $\mathbb{Z}_2$-grading, the natural bracket operation to use is the *supercommutator*, which is the bilinear map $\mathsf{T}(\mathfrak{g}) \times \mathsf{T}(\mathfrak{g}) \to \mathsf{T}(\mathfrak{g})$ defined by $\{a,b\} := ab - (-1)^{|a||b|} ba$ for homogeneous elements $a, b \in \mathsf{T}(\mathfrak{g})$. (For the time being, we use a different symbol $\{\,,\,\}$ for better distinction from the bracket $[\,,\,]$ on $\mathfrak{g}$.) Since by the definition of $\mathsf{J}(\mathfrak{g})$ one has

$$\{\mathsf{T}(\mathfrak{g}), \mathsf{J}(\mathfrak{g})\} = \{\mathsf{J}(\mathfrak{g}), \mathsf{T}(\mathfrak{g})\} \subset \mathsf{J}(\mathfrak{g})\,,$$

the supercommutator descends to a well-defined map $\{\,,\,\}: \mathsf{U}(\mathfrak{g}) \times \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$.

Lemma 2.2. — *If $\mathfrak{g}$ is a Lie superalgebra, the supercommutator $\{\,,\,\}$ gives $\mathsf{U}(\mathfrak{g})$ the structure of another Lie superalgebra in which $\{\mathsf{U}_n(\mathfrak{g}), \mathsf{U}_{n'}(\mathfrak{g})\} \subset \mathsf{U}_{n+n'-1}(\mathfrak{g})$.*

Proof. — Compatibility with the $\mathbb{Z}_2$-grading, skew symmetry, and Jacobi identity are properties of $\{\,,\}$ that are immediate at the level of the tensor algebra $\mathsf{T}(\mathfrak{g})$. They descend to the corresponding properties at the level of $\mathsf{U}(\mathfrak{g})$ by the definition of the two-sided ideal $\mathsf{J}(\mathfrak{g})$. Thus $\mathsf{U}(\mathfrak{g})$ with the bracket $\{\,,\}$ is a Lie superalgebra.

To see that $\{\mathsf{U}_n(\mathfrak{g}), \mathsf{U}_{n'}(\mathfrak{g})\}$ is contained in $\mathsf{U}_{n+n'-1}(\mathfrak{g})$, notice that this property holds true for $n = n' = 1$ by the defining relations $\mathsf{J}(\mathfrak{g}) \equiv 0$ of $\mathsf{U}(\mathfrak{g})$. Then use the associative law for $\mathsf{U}(\mathfrak{g})$ to verify the formula

$$\{a, bc\} = abc - (-1)^{|a|(|b|+|c|)} bca = \{a, b\}c + (-1)^{|a||b|} b\{a, c\}$$

for homogeneous $a, b, c \in \mathsf{U}(\mathfrak{g})$. The claim now follows by induction on the degree of the filtration $\mathsf{U}(\mathfrak{g}) = \cup_{n=0}^{\infty} \mathsf{U}_n(\mathfrak{g})$. □

By definition, the supercommutator of $\mathsf{U}(\mathfrak{g})$ and the bracket of $\mathfrak{g}$ agree at the linear level: $\{X, Y\} \equiv [X, Y]$ for $X, Y \in \mathfrak{g}$. It is therefore reasonable to drop the distinction in notation and simply write $[\,,]$ for both of these product operations. This we now do.

For future use, note the following variant of the preceding formula: if $Y_1, \ldots, Y_k, X$ are any homogeneous elements of $\mathfrak{g}$, then

$$[Y_1 \cdots Y_k\, , X] = \sum_{i=1}^{k} (-1)^{|X|\,\Sigma_{j=i+1}^{k} |Y_j|}\, Y_1 \cdots Y_{i-1}\, [Y_i\, , X]\, Y_{i+1} \cdots Y_k\ , \tag{2.2}$$

which expresses the supercommutator in $\mathsf{U}(\mathfrak{g})$ by the bracket in $\mathfrak{g}$.

2.2. Structure of $\mathfrak{osp}(W)$. — For a $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space $V = V_0 \oplus V_1$ let $W = V \oplus V^* = W_0 \oplus W_1$ as in Example 2.3. The *orthogonal Lie algebra* $\mathfrak{o}(W_1)$ is the Lie algebra of the Lie group $\mathrm{O}(W_1)$ of $\mathbb{K}$-linear transformations of W_1 that leave the non-degenerate symmetric bilinear form S invariant. This means that $X \in \mathrm{End}(W_1)$ is in $\mathfrak{o}(W_1)$ if and only if

$$\forall w, w' \in W_1 :\ S(Xw, w') + S(w, Xw') = 0\ .$$

Similarly, the *symplectic Lie algebra* $\mathfrak{sp}(W_0)$ is the Lie algebra of the automorphism group $\mathrm{Sp}(W_0)$ of W_0 equipped with the non-degenerate alternating bilinear form A :

$$\mathfrak{sp}(W_0) = \{X \in \mathrm{End}(W_0) \mid \forall w, w' \in W_0 :\ A(Xw, w') + A(w, Xw') = 0\}\ .$$

For the next statement, recall the definition of the orthosymplectic Lie superalgebra $\mathfrak{osp}(W)$ and the decomposition $\mathfrak{osp}(W) = \mathfrak{osp}(W)_0 \oplus \mathfrak{osp}(W)_1$.

Lemma 2.3. — *As Lie algebras resp. vector spaces,*

$$\mathfrak{osp}(W)_0 \simeq \mathfrak{o}(W_1) \oplus \mathfrak{sp}(W_0)\ , \quad \mathfrak{osp}(W)_1 \simeq W_1 \otimes W_0^*\ .$$

Proof. — The first isomorphism follows directly from the definitions. For the second isomorphism, decompose $X \in \mathfrak{osp}(W)_1$ as $X = X_{01} + X_{10}$ where $X_{01} \in \mathrm{Hom}(W_1, W_0)$ and $X_{10} \in \mathrm{Hom}(W_0, W_1)$. Then

$$\mathfrak{osp}(W)_1 = \{X_{10} + X_{01} \in \mathrm{End}(W)_1 \mid \forall w_s \in W_s : S(X_{10}w_0\, , w_1) + A(w_0\, , X_{01}w_1) = 0\}\ .$$

Since both S and A are non-degenerate, the component X_{01} is determined by the component X_{10}, and one therefore has $\mathfrak{osp}(W)_1 \simeq \mathrm{Hom}(W_0\, , W_1) \simeq W_1 \otimes W_0^*$. □

We now review how $\mathfrak{sp}(W_0)$ and $\mathfrak{o}(W_1)$ decompose for our case $W_s = V_s \oplus V_s^*$. For that purpose, if U is a vector space with dual vector space U^*, let $\mathrm{Sym}(U,U^*)$ and $\mathrm{Alt}(U,U^*)$ denote the symmetric resp. alternating linear maps from U to U^*.

Lemma 2.4. — *As vector spaces,*

$$\mathfrak{o}(W_1) \simeq \mathrm{End}(V_1) \oplus \mathrm{Alt}(V_1, V_1^*) \oplus \mathrm{Alt}(V_1^*, V_1) ,$$
$$\mathfrak{sp}(W_0) \simeq \mathrm{End}(V_0) \oplus \mathrm{Sym}(V_0, V_0^*) \oplus \mathrm{Sym}(V_0^*, V_0) .$$

Proof. — There is a canonical decomposition

$$\mathrm{End}(W_s) = \mathrm{End}(V_s) \oplus \mathrm{Hom}(V_s^*, V_s) \oplus \mathrm{Hom}(V_s, V_s^*) \oplus \mathrm{End}(V_s^*)$$

for $s = 0,1$. Let $s = 1$ and write the corresponding decomposition of $X \in \mathrm{End}(W_1)$ as

$$X = \mathsf{A} \oplus \mathsf{B} \oplus \mathsf{C} \oplus \mathsf{D} .$$

Substituting $w = v + \varphi$ and $w' = v' + \varphi'$, the defining condition $S(Xw, w') = -S(w, Xw')$ for $X \in \mathfrak{o}(W_1)$ then transcribes to

$$\varphi'(\mathsf{A}v) = -(\mathsf{D}\varphi')(v) , \quad (\mathsf{C}v)(v') = -(\mathsf{C}v')(v) , \quad \varphi'(\mathsf{B}\varphi) = -\varphi(\mathsf{B}\varphi') ,$$

for all $v, v' \in V_1$ and $\varphi, \varphi' \in V_1^*$. Thus $\mathsf{D} = -\mathsf{A}^{\mathrm{t}}$, and the maps B, C are alternating. This already proves the statement for the case of $\mathfrak{o}(W_1)$.

The situation for $\mathfrak{sp}(W_0)$ is identical but for a sign change: the symmetric form S is replaced by the alternating form A, and this causes the parity of B, C to be reversed. ☐

By adding up dimensions, Lemmas 2.3 and 2.4 entail the following consequence.

Corollary 2.1. — *As a $\mathbb{Z}_2$-graded vector space, $\mathfrak{osp}(V \oplus V^*)$ is isomorphic to $\mathbb{K}^{p|q}$ where $p = d_0(2d_0+1) + d_1(2d_1-1)$, $q = 4d_0 d_1$, and $d_s = \dim V_s$.*

There exists another way of thinking about $\mathfrak{osp}(W)$, which will play a key role in the sequel. To define it and keep the sign factors consistent and transparent, we need to be meticulous about our ordering conventions. Hence, if $v \in V$ is a vector and $\varphi \in V^*$ is a linear function, we write the value of φ on v as

$$\varphi(v) \equiv \langle v, \varphi \rangle .$$

Based on this notational convention, if V is a $\mathbb{Z}_2$-graded vector space and $X \in \mathrm{End}(V)$ is a homogeneous operator, we define the *supertranspose* $X^{\mathrm{st}} \in \mathrm{End}(V^*)$ of X by

$$\langle v, X^{\mathrm{st}} \varphi \rangle := (-1)^{|X||v|} \langle Xv, \varphi \rangle \quad (v \in V_0 \cup V_1,\ \varphi \in V^*) .$$

This definition differs from the usual transpose by a change of sign in the case when X has a component in $\mathrm{Hom}(V_1, V_0)$. From it, it follows directly that the negative supertranspose $\mathfrak{gl}(V) \to \mathfrak{gl}(V^*)$, $X \mapsto -X^{\mathrm{st}}$ is an isomorphism of Lie superalgebras:

$$-[X,Y]^{\mathrm{st}} = [-X^{\mathrm{st}}, -Y^{\mathrm{st}}] .$$

The modified notion of transpose goes hand in hand with a modified notion of what it means for an operator in $\mathrm{Hom}(V, V^*)$ or $\mathrm{Hom}(V^*, V)$ to be symmetric. Thus, define

the subspace $\mathrm{Sym}(V^*,V) \subset \mathrm{Hom}(V^*,V)$ to consist of the elements, say B, which are symmetric in the $\mathbb{Z}_2$-graded sense:

$$\forall\, \varphi,\varphi' \in V_0^* \cup V_1^* : \quad \langle \mathsf{B}\varphi,\varphi'\rangle = \langle \mathsf{B}\varphi',\varphi\rangle\,(-1)^{|\varphi||\varphi'|}\,. \tag{2.3}$$

By the same principle, define $\mathrm{Sym}(V,V^*) \subset \mathrm{Hom}(V,V^*)$ as the set of solutions C of

$$\forall\, v,v' \in V_0 \cup V_1 : \quad \langle v,\mathsf{C}v'\rangle = \langle v',\mathsf{C}v\rangle\,(-1)^{|v||v'|+|v|+|v'|}\,. \tag{2.4}$$

To make the connection with the decomposition of Lemma 2.3 and 2.4, notice that

$$\mathrm{Sym}(V,V^*) \cap \mathrm{Hom}(V_s,V_s^*) = \begin{cases} \mathrm{Sym}(V_0,V_0^*) & s=0\,, \\ \mathrm{Alt}(V_1,V_1^*) & s=1\,, \end{cases}$$

and similar for the corresponding intersections involving $\mathrm{Sym}(V^*,V)$.

Next, expressing the orthosymplectic form Q of $W = V \oplus V^*$ as

$$Q(v+\varphi, v'+\varphi') = \langle v,\varphi'\rangle - (-1)^{|v'||\varphi|}\langle v',\varphi\rangle\,,$$

and writing out the conditions resulting from $Q(Xw,w') + (-1)^{|X||w|}Q(w,Xw') = 0$ for the case of $X \equiv \mathsf{B} \in \mathrm{Hom}(V^*,V)$ and $X \equiv \mathsf{C} \in \mathrm{Hom}(V,V^*)$, one sees that

$$\mathfrak{osp}(W) \cap \mathrm{Hom}(V,V^*) = \mathrm{Sym}(V,V^*)\,, \quad \mathfrak{osp}(W) \cap \mathrm{Hom}(V^*,V) = \mathrm{Sym}(V^*,V)\,.$$

This situation is summarized in the next statement.

Lemma 2.5. — *The orthosymplectic Lie superalgebra of $W = V \oplus V^*$ decomposes as*

$$\mathfrak{osp}(W) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+2)}\,,$$

where $\mathfrak{g}^{(+2)} := \mathrm{Sym}(V,V^*)$, *and* $\mathfrak{g}^{(-2)} := \mathrm{Sym}(V^*,V)$, *and*

$$\mathfrak{g}^{(0)} := (\mathrm{End}(V) \oplus \mathrm{End}(V^*)) \cap \mathfrak{osp}(W)\,.$$

The decomposition of Lemma 2.5 can be regarded as a $\mathbb{Z}$-grading of $\mathfrak{osp}(W)$. By the 'block' structure inherited from $W = V \oplus V^*$, this decomposition is compatible with the bracket $[\,,]$:

$$[\mathfrak{g}^{(m)}, \mathfrak{g}^{(m')}] \subset \mathfrak{g}^{(m+m')}\,,$$

where $\mathfrak{g}^{(m+m')} \equiv 0$ if $m+m' \notin \{\pm 2, 0\}$. It follows that each of the three subspaces $\mathfrak{g}^{(+2)}$, $\mathfrak{g}^{(-2)}$, and $\mathfrak{g}^{(0)}$ is a Lie superalgebra, the first two with vanishing bracket.

Lemma 2.6. — *The embedding* $\mathrm{End}(V) \to \mathrm{End}(V) \oplus \mathrm{End}(V^*)$ *by* $\mathsf{A} \mapsto \mathsf{A} \oplus (-\mathsf{A}^{\mathrm{st}})$ *projected to* $\mathfrak{osp}(W)$ *is an isomorphism of Lie superalgebras* $\mathfrak{gl}(V) \to \mathfrak{g}^{(0)}$.

Proof. — Since the negative supertranspose $\mathsf{A} \mapsto -\mathsf{A}^{\mathrm{st}}$ is a homomorphism of Lie superalgebras, so is our embedding $\mathsf{A} \mapsto \mathsf{A} \oplus (-\mathsf{A}^{\mathrm{st}})$. This map is clearly injective. To see that it is surjective, consider any homogeneous $X = \mathsf{A} \oplus \mathsf{D} \in \mathrm{End}(V) \oplus \mathrm{End}(V^*)$ viewed as an operator in $\mathrm{End}(W)$. The condition for X to be in $\mathfrak{osp}(W)$ is (2.1). To get a non-trivial condition, choose $(w,w') = (v,\varphi)$ or $(w,w') = (\varphi,v)$. The first choice gives

$$Q(Xv,\varphi) + (-1)^{|X||v|}Q(v,X\varphi) = \langle \mathsf{A}v,\varphi\rangle + (-1)^{|\mathsf{A}||v|}\langle v,\mathsf{D}\varphi\rangle = 0\,.$$

Valid for all $v \in V_0 \cup V_1$ and $\varphi \in V^*$, this implies that $\mathsf{D} = -\mathsf{A}^{\mathrm{st}}$. The second choice leads to the same conclusion. Thus $X = \mathsf{A} \oplus \mathsf{D}$ is in $\mathfrak{osp}(W)$ if and only if $\mathsf{D} = -\mathsf{A}^{\mathrm{st}}$. $\square$

In the following subsections we will often write $\mathfrak{osp}(W) \equiv \mathfrak{osp}$ for short.

2.2.1. Roots and root spaces. — A Cartan subalgebra of a Lie algebra $\mathfrak{g}_0$ is a maximal commutative subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ such that $\mathfrak{g}_0$ (or its complexification if $\mathfrak{g}_0$ is a real Lie algebra) has a basis consisting of eigenvectors of $\mathrm{ad}(H)$ for all $H \in \mathfrak{h}$. Recall that $|[X,Y]| = |X| + |Y|$ for homogeneous elements X,Y of a Lie superalgebra $\mathfrak{g}$. From the vantage point of decomposing $\mathfrak{g}$ by eigenvectors or root spaces, it is therefore reasonable to call a Cartan subalgebra of $\mathfrak{g}_0$ a Cartan subalgebra of $\mathfrak{g}$. We will see that $X \in \mathfrak{osp}_1$ and $[X,H] = 0$ for all $H \in \mathfrak{h} \subset \mathfrak{osp}_0$ imply $X = 0$, i.e., there exists no commutative subalgebra of $\mathfrak{osp}$ that properly contains a Cartan subalgebra. Lie superalgebras with this property are called of type I in [**1**].

Let us determine a Cartan subalgebra and the corresponding root space decomposition of $\mathfrak{osp}$. For $s,t = 0,1$ choose bases $\{e_{s,1}, \ldots, e_{s,d_s}\}$ of V_s and associated dual bases $\{f_{t,1}, \ldots, f_{t,d_t}\}$ of V_t^*. Then for $j = 1, \ldots, d_s$ and $k = 1, \ldots, d_t$ define rank-one operators $E_{s,j;t,k}$ by the equation $E_{s,j;t,k}(e_{u,l}) = e_{s,j}\, \delta_{t,u}\, \delta_{k,l}$ for all $u = 0,1$ and $l = 1, \ldots, d_u$. These form a basis of $\mathrm{End}(V)$, and by Lemma 2.6 the operators

$$X^{(0)}_{sj,tk} := E_{s,j;t,k} \oplus (-E_{s,j;t,k})^{\mathrm{st}}$$

form a basis of $\mathfrak{g}^{(0)}$. Similarly, let bases of $\mathrm{Hom}(V^*,V)$ and $\mathrm{Hom}(V,V^*)$ be defined by

$$F_{s,j;t,k}(f_{u,l}) = e_{s,j}\, \delta_{t,u}\, \delta_{k,l}\ , \qquad \tilde{F}_{s,j;t,k}(e_{u,l}) = f_{s,j}\, \delta_{t,u}\, \delta_{k,l}\ ,$$

for index pairs in the appropriate range. Then by Lemma 2.5 and equations (2.3, 2.4) the subalgebras $\mathfrak{g}^{(-2)}$ and $\mathfrak{g}^{(2)}$ are generated by the sets of operators

$$\begin{aligned} X^{(-2)}_{sj,tk} &:= F_{s,j;t,k} + F_{t,k;s,j}\, (-1)^{|s||t|}\ , \\ X^{(2)}_{sj,tk} &:= \tilde{F}_{s,j;t,k} + \tilde{F}_{t,k;s,j}\, (-1)^{|s||t|+|s|+|t|}\ . \end{aligned}$$

Since $\mathfrak{osp}_0 \simeq \mathfrak{o}(W_1) \oplus \mathfrak{sp}(W_0)$, a Cartan subalgebra of $\mathfrak{osp}$ is the direct sum of a Cartan subalgebra of $\mathfrak{o}(W_1)$ and a Cartan subalgebra of $\mathfrak{sp}(W_0)$. Letting $\mathfrak{h}$ be the span of the diagonal operators

$$H_{sj} := X^{(0)}_{sj,sj} \quad (s = 0,1;\ j = 1, \ldots, d_s)\ ,$$

one has that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{osp}$. Indeed, if $\{\vartheta_{sj}\}$ denotes the basis of $\mathfrak{h}^*$ dual to $\{H_{sj}\}$, inspection of the adjoint action of $\mathfrak{h}$ on $\mathfrak{osp}$ gives the following result.

Lemma 2.7. — *The operators $X^{(m)}_{sj,tk}$ are eigenvectors of $\mathrm{ad}(H)$ for all $H \in \mathfrak{h}$:*

$$[H\,, X^{(m)}_{sj,tk}] = \begin{cases} (\vartheta_{sj} - \vartheta_{tk})(H)\, X^{(m)}_{sj,tk} & m = 0\ , \\ (\vartheta_{sj} + \vartheta_{tk})(H)\, X^{(m)}_{sj,tk} & m = -2\ , \\ (-\vartheta_{sj} - \vartheta_{tk})(H)\, X^{(m)}_{sj,tk} & m = 2\ . \end{cases}$$

A root of a Lie superalgebra $\mathfrak{g}$ is called *even* if its root space is in $\mathfrak{g}_0$, it is called *odd* if its root space is in $\mathfrak{g}_1$. We denote by Δ_0 and Δ_1 the set of even roots and the set of odd roots, respectively. For $\mathfrak{g} = \mathfrak{osp}$ we have

$$\Delta_0 = \{\pm\vartheta_{1j} \pm \vartheta_{1k}, \pm\vartheta_{0j} \pm \vartheta_{0l} \mid j < k, j \le l\}, \quad \Delta_1 = \{\pm\vartheta_{1j} \pm \vartheta_{0k}\}.$$

2.2.2. Casimir elements. — As before, let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra, and let $\mathsf{U}(\mathfrak{g}) = \cup_{n=0}^{\infty} \mathsf{U}_n(\mathfrak{g})$ be its universal enveloping algebra. Denote the symmetric algebra of $\mathfrak{g}_0$ by $\mathsf{S}(\mathfrak{g}_0)$ and the exterior algebra of $\mathfrak{g}_1$ by $\wedge(\mathfrak{g}_1)$. The Poincaré-Birkhoff-Witt theorem for Lie superalgebras states that for each n there is a bijective correspondence

$$\mathsf{U}_n(\mathfrak{g})/\mathsf{U}_{n-1}(\mathfrak{g}) \xrightarrow{\sim} \textstyle\sum_{k+l=n} \wedge^k(\mathfrak{g}_1) \otimes \mathsf{S}^l(\mathfrak{g}_0) .$$

The collection of inverse maps lift to a vector-space isomorphism,

$$\wedge(\mathfrak{g}_1) \otimes \mathsf{S}(\mathfrak{g}_0) \xrightarrow{\sim} \mathsf{U}(\mathfrak{g}) ,$$

called the super-symmetrization mapping. In other words, given a homogeneous basis $\{e_1, \ldots, e_d\}$ of $\mathfrak{g}$, each element $x \in \mathsf{U}(\mathfrak{g})$ can be uniquely represented in the form $x = \sum_n \sum_{i_1,\ldots,i_n} x_{i_1,\ldots,i_n} e_{i_1} \cdots e_{i_n}$ with super-symmetrized coefficients, i.e.,

$$x_{i_1,\ldots,i_l,i_{l+1},\ldots,i_n} = (-1)^{|e_{i_l}||e_{i_{l+1}}|} x_{i_1,\ldots,i_{l+1},i_l,\ldots,i_n} \quad (1 \le l < n) .$$

The isomorphism $\wedge(\mathfrak{g}_1) \otimes \mathsf{S}(\mathfrak{g}_0) \simeq \mathsf{U}(\mathfrak{g})$ gives $\mathsf{U}(\mathfrak{g})$ a $\mathbb{Z}$-grading (by the degree n).

Now recall that $\mathsf{U}(\mathfrak{g})$ comes with a canonical bracket operation, the supercommutator $[\,,\,] : \mathsf{U}(\mathfrak{g}) \times \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$. An element $X \in \mathsf{U}(\mathfrak{g})$ is said to lie in the center of $\mathsf{U}(\mathfrak{g})$, and is called a *Casimir element*, iff $[X,Y] = 0$ for all $Y \in \mathsf{U}(\mathfrak{g})$. By the formula (2.2), a necessary and sufficient condition for that is $[X,Y] = 0$ for all $Y \in \mathfrak{g}$.

In the case of $\mathfrak{g} = \mathfrak{osp}$, for every $\ell \in \mathbb{N}$ there is a Casimir element I_ℓ of degree 2ℓ, which is constructed as follows. Consider the bilinear form $B : \mathfrak{osp} \times \mathfrak{osp} \to \mathbb{K}$ given by the supertrace (in some representation), $B(X,Y) := \mathrm{STr}\,(XY)$. Recall that this form is ad-invariant, which is to say that $B([X,Y],Z) = B(X,[Y,Z])$ for all $X,Y,Z \in \mathfrak{g}$.

Taking the supertrace in the fundamental representation of $\mathfrak{osp}$, the form B is non-degenerate, and therefore, if $e_1, \ldots, e_d$ is a homogeneous basis of $\mathfrak{osp}$, there is another homogeneous basis $\widetilde{e}_1, \ldots, \widetilde{e}_d$ of $\mathfrak{osp}$ so that $B(\widetilde{e}_i, e_j) = \delta_{ij}$. Note $|\widetilde{e}_i| = |e_i|$ and put

$$I_\ell := \sum_{i_1,\ldots,i_{2\ell}=1}^{d} \widetilde{e}_{i_1} \cdots \widetilde{e}_{i_{2\ell}} \, \mathrm{STr}\,(e_{i_{2\ell}} \cdots e_{i_1}) \in \mathsf{U}(\mathfrak{osp}). \tag{2.5}$$

(Notice that, in view of the remark following Lemma 2.1, there is no point in making the same construction with an odd number of factors.)

Lemma 2.8. — *For all $\ell \in \mathbb{N}$ the element I_ℓ is Casimir, and $|I_\ell| = 0$.*

Proof. — By specializing the formula (2.2) to the present case,

$$[I_\ell, X] = \sum \sum_{k=1}^{2\ell} (-1)^{|X|(|\widetilde{e}_{i_{k+1}}|+\ldots+|\widetilde{e}_{i_{2\ell}}|)} \widetilde{e}_{i_1} \cdots \widetilde{e}_{i_{k-1}} [\widetilde{e}_{i_k}, X] \, \widetilde{e}_{i_{k+1}} \cdots \widetilde{e}_{i_{2\ell}} \, \mathrm{STr}\,(e_{i_{2\ell}} \cdots e_{i_1}) .$$

Now if $[\widetilde{e}_i, X] = \sum_j X_{ij}\,\widetilde{e}_j$ then from ad-invariance, $B([\widetilde{e}_i, X], e_j) = X_{ij} = B(\widetilde{e}_i, [X, e_j])$, one has $[X, e_j] = \sum_i e_i X_{ij}$. Using this relation to transfer the ad(X)-action from $\widetilde{e}_{i_k}$ to e_{i_k}, and reading the formula (2.2) backwards, one obtains

$$[I_\ell\, , X] = \sum \widetilde{e}_{i_1} \cdots \widetilde{e}_{i_{2\ell}}\, \mathrm{STr}\,([X, e_{i_{2\ell}} \cdots e_{i_1}]) \; .$$

Since the supertrace of any bracket vanishes, one concludes that $[I_\ell\, , X] = 0$.

The other statement, $|I_\ell| = 0$, follows from $|\widetilde{e}_i| = |e_i|$, the additivity of the $\mathbb{Z}_2$-degree and the fact that $\mathrm{STr}\,(a) = 0$ for any odd element $a \in \mathsf{U}(\mathfrak{g})$. ☐

We now describe a useful property enjoyed by the Casimir elements I_ℓ of $\mathfrak{osp}(V \oplus V^*)$ in the special case of isomorphic components $V_0 \simeq V_1$. Recalling the notation of §2.2.1, let $\partial := \sum_j X^{(0)}_{0j,1j}$ and $\widetilde{\partial} := -\sum_j X^{(0)}_{1j,0j}$. These are odd elements of $\mathfrak{osp}$. (The reason for using the symbols $\partial, \widetilde{\partial}$ will become clear later). Notice that the bracket $\Lambda := [\partial, \widetilde{\partial}] = -\sum_{s,j} H_{sj}$ is in the Cartan algebra of $\mathfrak{osp}$. From $[\partial, \partial] = 2\,\partial^2 = 0$ and the Jacobi identity one infers that

$$[\partial, \Lambda] = [[\partial, \partial], \widetilde{\partial}] - [\partial, [\partial, \widetilde{\partial}]] = -[\partial, \Lambda] = 0 \; .$$

By the same argument, $[\widetilde{\partial}, \Lambda] = 0$. One also sees that $\Lambda^2 = \mathrm{Id}$.

Now define F_ℓ to be the following odd element of $\mathsf{U}(\mathfrak{osp})$:

$$F_\ell = -\sum_{i_1,\ldots,i_{2\ell}=1}^{d} \widetilde{e}_{i_1} \cdots \widetilde{e}_{i_{2\ell}}\, \mathrm{STr}\,(e_{i_{2\ell}} \cdots e_{i_1} \widetilde{\partial} \Lambda) \; .$$

Lemma 2.9. — *Let $\mathfrak{osp}(V \oplus V^*)$ be the orthosymplectic Lie superalgebra for a $\mathbb{Z}_2$-graded vector space V with isomorphic components $V_0 \simeq V_1$. Then for all $\ell \in \mathbb{N}$ the Casimir element I_ℓ is expressible as a bracket: $I_\ell = [\partial, F_\ell]$.*

Proof. — By the same argument as in the proof of Lemma 2.8,

$$[\partial, F_\ell] = -\sum \widetilde{e}_{i_1} \cdots \widetilde{e}_{i_{2\ell}}\, \mathrm{STr}\,([\partial, e_{i_{2\ell}} \cdots e_{i_1}]\, \widetilde{\partial} \Lambda) \; .$$

Using the relations $[\partial, \Lambda] = 0$ and $\Lambda^2 = \mathrm{Id}$, one has for any $a \in \mathsf{U}(\mathfrak{osp})$ that

$$-\mathrm{STr}\,([\partial, a]\, \widetilde{\partial} \Lambda) = \mathrm{STr}\,(\widetilde{\partial} \Lambda\, [\partial, a]) = \mathrm{STr}\,([\widetilde{\partial}, \partial] \Lambda\, a) = \mathrm{STr}\,(\Lambda^2 a) = \mathrm{STr}\,(a) \; ,$$

where the second equality sign is from $\mathrm{STr}\,(c, [b, a]) = \mathrm{STr}\,([c, b]\, a)$. The statement of the lemma now follows on setting $a = e_{i_{2\ell}} \cdots e_{i_1}$. ☐

As we shall see, Lemma 2.9 has the drastic consequence that all $\mathfrak{osp}$-Casimir elements I_ℓ are zero in the spinor-oscillator representation of $\mathfrak{osp}(V \oplus V^*)$ for $V_0 \simeq V_1$.

2.3. Howe pairs in $\mathfrak{osp}$. — In the present context, a pair $(\mathfrak{h}, \mathfrak{h}')$ of subalgebras $\mathfrak{h}, \mathfrak{h}' \subset \mathfrak{g}$ of a Lie superalgebra $\mathfrak{g}$ is called a *dual pair* whenever $\mathfrak{h}'$ is the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ and vice versa. In this subsection, let $\mathbb{K} = \mathbb{C}$.

Given a $\mathbb{Z}_2$-graded complex vector space $U = U_0 \oplus U_1$ we let $V := U \otimes \mathbb{C}^N$, where $\mathbb{C}^N$ is equipped with the standard representation of $\mathrm{GL}(\mathbb{C}^N)$, $\mathrm{O}(\mathbb{C}^N)$, or $\mathrm{Sp}(\mathbb{C}^N)$, as the case may be. As a result, the Lie algebra $\mathfrak{k}$ of whichever group is represented on

$\mathbb{C}^N$ is embedded in $\mathfrak{osp}(V \oplus V^*)$. We will now describe the dual pairs $(\mathfrak{h}, \mathfrak{k})$ in $\mathfrak{osp}(W)$ for $W = V \oplus V^*$. These are known as dual pairs in the sense of R. Howe.

Let us begin by recalling that for any representation $\rho : K \to \mathrm{GL}(E)$ of a group K on a vector space E, the dual representation $\rho^* : K \to \mathrm{GL}(E^*)$ on the linear forms on E is given by $(\rho(k)\varphi)(x) = \varphi(\rho(k)^{-1}x)$. By this token, every representation $\rho : K \to \mathrm{GL}(\mathbb{C}^N)$ induces a representation $\rho_W = (\mathrm{Id} \otimes \rho) \times (\mathrm{Id} \otimes \rho^*)$ of K on $W = V \oplus V^*$.

Lemma 2.10. — *Let $\rho : K \to \mathrm{GL}(\mathbb{C}^N)$ be any representation of a Lie group K. If $V = U \otimes \mathbb{C}^N$ for a $\mathbb{Z}_2$-graded complex vector space $U = U_0 \oplus U_1$, the induced representation $\rho_{W*}(\mathfrak{k})$ of the Lie algebra $\mathfrak{k}$ of K on $W = V \oplus V^*$ is a subalgebra of $\mathfrak{osp}(W)_0$.*

Proof. — The K-action on $\mathbb{C}^N \otimes (\mathbb{C}^N)^*$ by $z \otimes \zeta \mapsto \rho(k)z \otimes \rho^*(k)\zeta$ preserves the canonical pairing $z \otimes \zeta \mapsto \zeta(z)$ between $\mathbb{C}^N$ and $(\mathbb{C}^N)^*$. Consequently, the K-action on $V \otimes V^*$ by $(\mathrm{Id} \otimes \rho) \otimes (\mathrm{Id} \otimes \rho^*)$ preserves the canonical pairing $V \otimes V^* \to \mathbb{C}$. Since the orthosymplectic form $Q : W \times W \to \mathbb{C}$ uses nothing but that pairing, it follows that

$$Q(\rho_W(k)w\,,\, \rho_W(k)w') = Q(w, w') \quad (\text{for all } w, w' \in W)\ .$$

Passing to the Lie algebra level one obtains $\rho_{W*}(\mathfrak{k}) \subset \mathfrak{osp}(W)$. The operator $\rho_W(k)$ preserves the $\mathbb{Z}_2$-grading of W; therefore one actually has $\rho_{W*}(\mathfrak{k}) \subset \mathfrak{osp}(W)_0$. $\square$

Let us now assume that the complex Lie group K is defined by a non-degenerate bilinear form $B : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ in the sense that

$$K = \{k \in \mathrm{GL}(\mathbb{C}^N) \mid \forall z, z' \in \mathbb{C}^N : B(kz\,, kz') = B(z\,, z')\}\ .$$

We then have a canonical isomorphism $\psi : \mathbb{C}^N \to (\mathbb{C}^N)^*$ by $z \mapsto B(z\,, \,)$, and an isomorphism $\Psi : (U \oplus U^*) \otimes \mathbb{C}^N \to W$ by $(u + \varphi) \otimes z \mapsto u \otimes z + \varphi \otimes \psi(z)$.

Lemma 2.11. — $\rho_W(k) = \Psi \circ (\mathrm{Id} \otimes k) \circ \Psi^{-1}$ *for all* $k \in K$.

Proof. — If $u \in U$, $\varphi \in U^*$, and $z \in \mathbb{C}^N$, then by the definition of Ψ and $\rho_W(k)$,

$$\rho_W(k)\Psi((u + \varphi) \otimes z) = u \otimes kz + \varphi \otimes \psi(z)k^{-1}\ .$$

Since B is K-invariant, one has $\psi(z)k^{-1} = \psi(kz)$, and therefore

$$u \otimes kz + \varphi \otimes \psi(z)k^{-1} = u \otimes kz + \varphi \otimes \psi(kz) = \Psi((\mathrm{Id} \otimes k)((u + \varphi) \otimes z))\ .$$

Thus $\rho_W(k) \circ \Psi = \Psi \circ (\mathrm{Id} \otimes k)$. $\square$

Let us now examine what happens to the orthosymplectic form Q on W when it is pulled back by the isomorphism Ψ to a bilinear form Ψ^*Q on $(U \oplus U^*) \otimes \mathbb{C}^N$:

$$\Psi^*Q((u + \varphi) \otimes z\,, (u' + \varphi') \otimes z') = \varphi'(u)\, \psi(z')(z) - (-1)^{|u'||\varphi|} \varphi(u')\, \psi(z)(z')\ .$$

By definition, $\psi(z)(z') = B(z\,, z')$, and writing $B(z\,, z') = (-1)^\delta B(z', z)$ where $\delta = 0$ if B is symmetric and $\delta = 1$ if B is alternating, we obtain

$$\Psi^*Q((u + \varphi) \otimes z\,, (u' + \varphi') \otimes z') = (\varphi'(u) - (-1)^{|u'||\varphi| + \delta} \varphi(u'))B(z', z)\ . \tag{2.6}$$

In view of this, let $\widetilde{U}$ denote the vector space $U = U_0 \oplus U_1$ with the twisted $\mathbb{Z}_2$-grading, i.e. $\widetilde{U}_s := U_{s+1}$ ($s \in \mathbb{Z}_2$). Moreover, notice that Ψ determines an embedding

$$\mathrm{End}(U \oplus U^*) \otimes \mathrm{End}(\mathbb{C}^N) \to \mathrm{End}(W) , \quad X \otimes k \mapsto \Psi \circ (X \otimes k) \circ \Psi^{-1} ,$$

whose restriction to $\mathrm{End}(U \oplus U^*) \otimes \{\mathrm{Id}\} \to \mathrm{End}(W)$ is an injective homomorphism.

In the following we often write $\mathrm{O}(\mathbb{C}^N) \equiv \mathrm{O}_N$ and $\mathrm{Sp}(\mathbb{C}^N) \equiv \mathrm{Sp}_N$ for short.

Corollary 2.2. — *For $K = \mathrm{O}_N$ and $K = \mathrm{Sp}_N$, the map $X \mapsto \Psi \circ (X \otimes \mathrm{Id}) \circ \Psi^{-1}$ defines a Lie superalgebra embedding into $\mathfrak{osp}(W)$ of $\mathfrak{osp}(U \oplus U^*)$ resp. $\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)$.*

Proof. — For $K = \mathrm{O}_N$ the bilinear form B of $\mathbb{C}^N$ is symmetric and the bilinear form Q of W pulls back – see (2.6) – to the standard orthosymplectic form of $U \oplus U^*$.

For $K = \mathrm{Sp}_N$, the form B is alternating. Its pullback, the orthosymplectic form of $U \oplus U^*$ twisted by the sign factor $(-1)^\delta$, is restored to standard form by switching to the $\mathbb{Z}_2$-graded vector space $\widetilde{U} \oplus \widetilde{U}^*$ with the twisted $\mathbb{Z}_2$-grading. $\square$

To go further, we need a statement concerning $\mathrm{Hom}_G(V_1, V_2)$, the space of G-equivariant homomorphisms between two modules V_1 and V_2 for a group G.

Lemma 2.12. — *Let X_1, X_2, Y_1, Y_2 be finite-dimensional vector spaces all of which are representation spaces for a group G. If the G-action on X_1 and X_2 is trivial, then*

$$\mathrm{Hom}_G(X_1 \otimes Y_1, X_2 \otimes Y_2) \simeq \mathrm{Hom}(X_1, X_2) \otimes \mathrm{Hom}_G(Y_1, Y_2) .$$

Proof. — $\mathrm{Hom}(X_1 \otimes Y_1, X_2 \otimes Y_2)$ is canonically isomorphic to $X_1^* \otimes Y_1^* \otimes X_2 \otimes Y_2$ as a G-representation space, with G-equivariant maps corresponding to G-invariant tensors. Since the G-action on $X_1^* \otimes X_2$ is trivial, one sees that $\mathrm{Hom}_G(X_1 \otimes Y_1, X_2 \otimes Y_2)$ is isomorphic to the tensor product of $X_1^* \otimes X_2 \simeq \mathrm{Hom}(X_1, X_2)$ with the space of G-invariants in $Y_1^* \otimes Y_2$. The latter in turn is isomorphic to $\mathrm{Hom}_G(Y_1, Y_2)$. $\square$

Proposition 2.1. — *Writing $\mathfrak{g}_N \equiv \mathfrak{g}(\mathbb{C}^N)$ for $\mathfrak{g} = \mathfrak{gl}$, $\mathfrak{o}$, $\mathfrak{sp}$, the following pairs are dual pairs in $\mathfrak{osp}(W)$: $(\mathfrak{gl}(U), \mathfrak{gl}_N)$, $(\mathfrak{osp}(U \oplus U^*), \mathfrak{o}_N)$, $(\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*), \mathfrak{sp}_N)$.*

Proof. — Here we calculate the centralizer of $\mathfrak{k}$ in $\mathfrak{osp}(W)$ for each of the three cases $\mathfrak{k} = \mathfrak{gl}_N$, $\mathfrak{o}_N$, $\mathfrak{sp}_N$ and refer the reader to [**7**] for the remaining details.

Since both $V \subset W$ and $V^* \subset W$ are K-invariant subspaces, $\mathrm{End}_K(W)$ decomposes as

$$\mathrm{End}_K(W) = \mathrm{End}_K(V) \oplus \mathrm{Hom}_K(V^*, V) \oplus \mathrm{Hom}_K(V, V^*) \oplus \mathrm{End}_K(V^*) .$$

By Schur's lemma, $\mathrm{End}_K(\mathbb{C}^N) \simeq \mathbb{C}$, and therefore Lemma 2.12 implies

$$\mathrm{End}_K(V) = \mathrm{End}_K(U \otimes \mathbb{C}^N) \simeq \mathrm{End}(U) \otimes \mathrm{End}_K(\mathbb{C}^N) = \mathrm{End}(U) .$$

By the same reasoning, $\mathrm{End}_K(V^*) = \mathrm{End}(U^*)$. Applying Lemma 2.12 to the two remaining summands, we obtain

$$\mathrm{Hom}_K(V, V^*) \simeq \mathrm{Hom}(U, U^*) \otimes \mathrm{Hom}_K(\mathbb{C}^N, \mathbb{C}^{N*}),$$

plus the same statement where each vector space is replaced by its dual.

If $K = \mathrm{GL}(\mathbb{C}^N) \equiv \mathrm{GL}_N$, then $\mathrm{Hom}_K(\mathbb{C}^N, \mathbb{C}^{N*}) = \mathrm{Hom}_K(\mathbb{C}^{N*}, \mathbb{C}^N) = \{0\}$. Hence,

$$\Phi : \ \mathrm{End}(U) \oplus \mathrm{End}(U^*) \to \mathrm{End}_{\mathrm{GL}_N}(W) , \quad X \oplus Y \mapsto (X \otimes \mathrm{Id}) \times (Y \otimes \mathrm{Id}) ,$$

for $W = U \otimes \mathbb{C}^N \oplus U^* \otimes \mathbb{C}^{N*}$ is an isomorphism. This means that the centralizer of $\mathfrak{gl}_N$ in $\mathfrak{osp}(W)$ is the intersection $\Phi(\mathrm{End}(U) \oplus \mathrm{End}(U^*)) \cap \mathfrak{osp}(W)$, which can be identified with $\mathrm{End}(U) = \mathfrak{gl}(U)$. Thus we have the first dual pair, $(\mathfrak{gl}(U), \mathfrak{gl}_N)$.

In the case of $K = \mathrm{O}_N$, the discussion is shortened by recalling Lemma 2.11 and the K-equivariant isomorphism $\Psi : (U \oplus U^*) \otimes \mathbb{C}^N \to W$. By Schur's lemma, these imply $\mathrm{End}_K(W) \simeq \mathrm{End}(U \oplus U^*)$. From Corollary 2.2 it then follows that the intersection $\mathfrak{osp}(W) \cap \mathrm{End}_K(W)$ is isomorphic as a Lie superalgebra to $\mathfrak{osp}(U \oplus U^*)$. Passing to the Lie algebra level for K, we get the second dual pair, $(\mathfrak{osp}(U \oplus U^*), \mathfrak{o}_N)$.

Finally, if $K = \mathrm{Sp}_N$, the situation is identical except that Corollary 2.2 compels us to switch to the $\mathbb{Z}_2$-twisted structure of orthosymplectic Lie superalgebra in $\mathrm{End}_K(W) \simeq \mathrm{End}(U \oplus U^*)$. This gives us the third dual pair, $(\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*), \mathfrak{sp}_N)$. □

2.4. Clifford-Weyl algebra $\mathfrak{q}(W)$. — Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ (in this subsection the choice of number field again is immaterial) and recall from Example 2.4 the definition of the Jordan-Heisenberg Lie superalgebra $\widetilde{W} = W \oplus \mathbb{K}$, where $W = W_0 \oplus W_1$ is a $\mathbb{Z}_2$-graded vector space with components $W_1 = V_1 \oplus V_1^*$ and $W_0 = V_0 \oplus V_0^*$. The universal enveloping algebra of the Jordan-Heisenberg Lie superalgebra is called the *Clifford-Weyl algebra* (or quantum algebra). We denote it by $\mathfrak{q}(W) \equiv \mathsf{U}(\widetilde{W})$.

Equivalently, one defines the Clifford-Weyl algebra $\mathfrak{q}(W)$ as the associative algebra generated by $\widetilde{W} = W \oplus \mathbb{K}$ subject to the following relations for all $w, w' \in W_0 \cup W_1$:

$$ww' - (-1)^{|w||w'|} w'w = Q(w, w') \, .$$

In particular, $w_0 w_1 = w_1 w_0$ for all $w_0 \in W_0$ and $w_1 \in W_1$. Reordering by this commutation relation defines an isomorphism of associative algebras $\mathfrak{q}(W) \simeq \mathfrak{c}(W_1) \otimes \mathfrak{w}(W_0)$, where the Clifford algebra $\mathfrak{c}(W_1)$ is generated by $W_1 \oplus \mathbb{K}$ with the relations $ww' + w'w = S(w, w')$ for $w, w' \in W_1$, and the Weyl algebra $\mathfrak{w}(W_0)$ is generated by $W_0 \oplus \mathbb{K}$ with the relations $ww' - w'w = A(w, w')$ for $w, w' \in W_0$.

As a universal enveloping algebra the Clifford-Weyl algebra $\mathfrak{q}(W)$ is filtered,

$$\mathfrak{q}_0(W) := \mathbb{K} \subset \mathfrak{q}_1(W) := W \oplus \mathbb{K} \subset \ldots \subset \mathfrak{q}_n(W) \ldots \, ,$$

and it inherits from the Jordan-Heisenberg algebra $\widetilde{W}$ a canonical $\mathbb{Z}_2$-grading and a canonical structure of Lie superalgebra by the supercommutator – see §2.1.2 for the definitions. The next statement is a sharpened version of Lemma 2.2.

Lemma 2.13. — $[\mathfrak{q}_n(W), \mathfrak{q}_{n'}(W)] \subset \mathfrak{q}_{n+n'-2}(W)$.

Proof. — Lemma 2.2 asserts the commutation relation $[\mathsf{U}_n(\mathfrak{g}), \mathsf{U}_{n'}(\mathfrak{g})] \subset \mathsf{U}_{n+n'-1}(\mathfrak{g})$ for the general case of a Lie superalgebra $\mathfrak{g}$ with bracket $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$. For the specific case at hand, where the fundamental bracket $[W, W] \subset \mathbb{K}$ has zero component in W, the degree $n + n' - 1$ is lowered to $n + n' - 2$ by the very argument proving that lemma. □

It now follows that each of the subspaces $\mathfrak{q}_n(W)$ for $n \leq 2$ is a Lie superalgebra. Since $[\mathfrak{q}_2(W), \mathfrak{q}_1(W)] \subset \mathfrak{q}_1(W)$, the quotient space $\mathfrak{q}_2(W)/\mathfrak{q}_1(W)$ is also a Lie superalgebra. By the Poincaré-Birkhoff-Witt theorem, there exists a vector-space isomorphism

$$\mathfrak{q}_2(W)/\mathfrak{q}_1(W) \xrightarrow{\sim} \mathfrak{s} \, ,$$

sending $\mathfrak{q}_2(W)/\mathfrak{q}_1(W)$ to $\mathfrak{s}$, the space of super-symmetrized degree-two elements in $\mathfrak{q}_2(W)$. Hence $\mathfrak{q}_2(W)$ has a direct-sum decomposition $\mathfrak{q}_2(W) = \mathfrak{q}_1(W) \oplus \mathfrak{s}$.

If $\{e_i\}$ is a homogeneous basis of W, every $a \in \mathfrak{s}$ is uniquely expressed as

$$a = \sum\nolimits_{i,j} a_{ij}\, e_i\, e_j \ , \quad a_{ij} = (-1)^{|e_i||e_j|} a_{ji} \ . \tag{2.7}$$

By adding and subtracting terms,

$$2ww' = (ww' + (-1)^{|w||w'|} w'w) + (ww' - (-1)^{|w||w'|} w'w) \ ,$$

one sees that the product ww' for $w, w' \in W$ has scalar part $(ww')_{\mathbb{K}} = \frac{1}{2}[w,w'] = \frac{1}{2} Q(w,w')$ with respect to the decomposition $\mathfrak{q}_2(W) = \mathbb{K} \oplus W \oplus \mathfrak{s}$.

Lemma 2.14. — $[\mathfrak{s},\mathfrak{s}] \subset \mathfrak{s}$.

Proof. — From the definition of $\mathfrak{s}$ and $[W,W] \subset \mathbb{K}$ it is clear that $[\mathfrak{s},\mathfrak{s}] \subset \mathbb{K} \oplus \mathfrak{s}$. The statement to be proved, then, is that $[a,b]$ for $a,\, b \in \mathfrak{s}$ has zero scalar part.

By the linearity of the supercommutator, it suffices to consider a single term of the sum (2.7). Thus we put $a = ww' + (-1)^{|w||w'|} w'w$, and have

$$\tfrac{1}{2}[a,b] = [ww',b] = w\,[w',b] + [w,b]\,w'(-1)^{|w'||b|} \ .$$

Now we compute the scalar part of the right-hand side. Using the Jacobi identity for the Lie superalgebra $\mathfrak{q}(W)$ we obtain

$$[a,b]_{\mathbb{K}} = [w,[w',b]] + [[w,b],w'](-1)^{|w'||b|} = [[w,w'],b] \ .$$

The last expression vanishes because $[w,w'] \subset \mathbb{K}$ lies in the center of $\mathfrak{q}(W)$. ☐

2.5. $\mathfrak{osp}(W)$ **inside** $\mathfrak{q}(W)$**.** — As a subspace of $\mathfrak{q}(W)$ which closes w.r.t. the supercommutator $[\, ,\,]$, $\mathfrak{s}$ is a Lie superalgebra. Now from Lemma 2.13 and the Jacobi identity for $\mathfrak{q}(W)$, one sees that $\mathfrak{s} \subset \mathfrak{q}_2(W)$ acts on each of the quotient spaces $\mathfrak{q}_n(W)/\mathfrak{q}_{n-1}(W)$ for $n \geq 1$ by $a \mapsto [a,\ ]$. In particular, $\mathfrak{s}$ acts on $\mathfrak{q}_1(W)/\mathfrak{q}_0(W) = W$ by $a \mapsto [a,w]$, which defines a homomorphism of Lie superalgebras

$$\tau : \mathfrak{s} \to \mathfrak{gl}(W) \ , \quad a \mapsto \tau(a) = [a,\ ] \ .$$

The mapping τ is actually into $\mathfrak{osp}(W) \subset \mathfrak{gl}(W)$. Indeed, for $w, w' \in W$ one has

$$Q(\tau(a)w,w') + (-1)^{|\tau(a)||w|} Q(w,\tau(a)w') = [[a,w],w'] + (-1)^{|a||w|}[w,[a,w']] \ ,$$

and since $[a,[w,w']] = 0$, this vanishes by the Jacobi identity.

Lemma 2.15. — *The map $\tau : \mathfrak{s} \to \mathfrak{osp}(W)$ is an isomorphism of Lie superalgebras.*

Proof. — Being a homomorphism of Lie superalgebras, the linear mapping τ is an isomorphism of such algebras if it is bijective. We first show that τ is injective. So, let $a \in \mathfrak{s}$ be any element of the kernel of τ. The equation $\tau(a) = 0$ means that $[[a,w],w'] = [\tau(a)w,w']$ vanishes for all $w, w' \in W$. To fathom the consequences of this, let $\{e_i\}$ and $\{\widetilde{e}_i\}$ be two homogeneous bases of W so that $Q(e_i,\widetilde{e}_j) = \delta_{ij}$. Using that $a \in \mathfrak{s}$

has a uniquely determined expansion $a = \sum a_{ij}\, e_i\, e_j$ with supersymmetric coefficients $a_{ij} = (-1)^{|e_i||e_j|} a_{ji}$, one computes

$$[[a, \widetilde{e}_j], \widetilde{e}_i] = a_{ij} + (-1)^{|e_i||e_j|} a_{ji} = 2a_{ij} \; .$$

Thus the condition $[[a,w],w'] = 0$ for all $w, w' \in W$ implies $a = 0$. Hence τ is injective.

By the Poincaré-Birkhoff-Witt isomorphism

$$\mathfrak{s} \simeq \mathfrak{q}_2(W)/\mathfrak{q}_1(W) \simeq \sum\nolimits_{k+l=2} \wedge^k(W_1) \otimes \mathrm{S}^l(W_0) \; ,$$

the dimensions of the $\mathbb{Z}_2$-graded vector space $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$ are

$$\dim \mathfrak{s}_0 = \dim \wedge^2(W_1) + \dim \mathrm{S}^2(W_0) \; , \quad \dim \mathfrak{s}_1 = \dim W_1 \dim W_0 \; .$$

These agree with those of $\mathfrak{osp}(W)$ as recorded in Corollary 2.1. Hence our injective linear map $\tau : \mathfrak{s} \to \mathfrak{osp}(W)$ is in fact a bijection. □

Remark 2.1. — By the isomorphism τ every representation ρ of $\mathfrak{s} \subset \mathfrak{q}(W)$ induces a representation $\rho \circ \tau^{-1}$ of $\mathfrak{osp}(W)$.

Let us conclude this subsection by writing down an explicit formula for τ^{-1}. To do so, let $\{e_i\}$ and $\{\widetilde{e}_j\}$ be homogeneous bases of W with $Q(e_i\,, \widetilde{e}_j) = \delta_{ij}$ as before. For $X \in \mathfrak{osp}(W)$ notice that the coefficients $a_{ij} := Q(e_i\,, X e_j)(-1)^{|e_j|}$ are supersymmetric:

$$a_{ij} = (-1)^{|X||e_i|+1+|e_j|} Q(Xe_i\,, e_j) = (-1)^{|X||e_i|} Q(e_j\,, Xe_i) = (-1)^{|e_i||e_j|} a_{ji} \; ,$$

where the last equality sign uses $(-1)^{|e_i||e_j|}\, a_{ji} = (-1)^{|e_i||Xe_i|}\, a_{ji} = (-1)^{|e_i||X|+|e_i|}\, a_{ji}$.

The inverse map $\tau^{-1} : \mathfrak{osp}(W) \to \mathfrak{s}$ is now expressed as

$$\tau^{-1}(X) = \tfrac{1}{2} \sum\nolimits_{i,j} Q(e_i\,, X e_j)(-1)^{|e_j|+1}\, \widetilde{e}_i\, \widetilde{e}_j \; . \tag{2.8}$$

To verify this formula, one calculates the double supercommutator $[e_i\,, [\tau^{-1}(X), e_j]]$ and shows that the result is equal to $[e_i\,, X e_j] = Q(e_i\,, X e_j)$, which is precisely what is required from the definition of τ by $[\tau^{-1}(X), e_j] = X e_j$.

2.6. Spinor-oscillator representation. — As before, starting from a $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space $V = V_0 \oplus V_1$, let the direct sum $W = V \oplus V^*$ be equipped with the orthosymplectic form Q and denote by $\mathfrak{q}(W)$ the Clifford-Weyl algebra of W.

Consider now the following tensor product of exterior and symmetric algebras:

$$\mathfrak{a}(V) := \wedge(V_1^*) \otimes \mathrm{S}(V_0^*) \; .$$

Following R. Howe we call it the *spinor-oscillator module* of $\mathfrak{q}(W)$. Notice that $\mathfrak{a}(V)$ can be identified with the graded-commutative subalgebra in $\mathfrak{q}(W)$ which is generated by $V \oplus \mathbb{K}$. As such, $\mathfrak{a}(V)$ comes with a canonical $\mathbb{Z}_2$-grading and its space of endomorphisms carries a structure of Lie superalgebra, $\mathfrak{gl}(\mathfrak{a}(V)) \equiv \mathrm{End}(\mathfrak{a}(V))$.

The algebra $\mathfrak{a}(V)$ now is to become a representation space for $\mathfrak{q}(W)$. Four operations are needed for this: the operator $\varepsilon(\varphi_1) : \wedge^k(V_1^*) \to \wedge^{k+1}(V_1^*)$ of exterior multiplication by a linear form $\varphi_1 \in V_1^*$; the operator $\iota(v_1) : \wedge^k(V_1^*) \to \wedge^{k-1}(V_1^*)$ of alternating

contraction with a vector $v_1 \in V_1$; the operator $\mu(\varphi_0) : \mathrm{S}^l(V_0^*) \to \mathrm{S}^{l+1}(V_0^*)$ of multiplication with a linear function $\varphi_0 \in V_0^*$; and the operator $\delta(v_0) : \mathrm{S}^l(V_0^*) \to \mathrm{S}^{l-1}(V_0^*)$ of taking the directional derivative by a vector $v_0 \in V_0$.

The operators ε and ι obey the *canonical anti-commutation relations* (CAR), which is to say that $\varepsilon(\varphi)$ and $\varepsilon(\varphi')$ anti-commute, $\iota(v)$ and $\iota(v')$ do as well, and one has

$$\iota(v)\varepsilon(\varphi) + \varepsilon(\varphi)\iota(v) = \varphi(v)\,\mathrm{Id}_{\wedge(V_1^*)} .$$

The operators μ and δ obey the *canonical commutation relations* (CCR), i.e., $\mu(\varphi)$ and $\mu(\varphi')$ commute, so do $\delta(v)$ and $\delta(v')$, and one has

$$\delta(v)\mu(\varphi) - \mu(\varphi)\delta(v) = \varphi(v)\,\mathrm{Id}_{\mathrm{S}(V_0^*)} .$$

Given all these operations, one defines a linear mapping $q : W \to \mathrm{End}(\mathfrak{a}(V))$ by

$$q(v_1 + \varphi_1 + v_0 + \varphi_0) = \iota(v_1) + \varepsilon(\varphi_1) + \delta(v_0) + \mu(\varphi_0) \quad (v_s \in V_s\,,\, \varphi_s \in V_s^*) ,$$

with $\iota(v_1)$, $\varepsilon(\varphi_1)$ operating on the first factor of the tensor product $\wedge(V_1^*) \otimes \mathrm{S}(V_0^*)$, and $\delta(v_0)$, $\mu(\varphi_0)$ on the second factor. Of course the two sets ε, ι and μ, δ commute with each other. In terms of q, the relations CAR and CCR are succinctly summarized as

$$[q(w)\,,\, q(w')] = Q(w,w')\,\mathrm{Id}_{\mathfrak{a}(V)} , \tag{2.9}$$

where $[\,,\,]$ denotes the usual supercommutator of the Lie superalgebra $\mathfrak{gl}(\mathfrak{a}(V))$. By the relation (2.9) the linear map q extends to a representation of the Jordan-Heisenberg Lie superalgebra $\widetilde{W} = W \oplus \mathbb{K}$, with the constants of $\widetilde{W}$ acting as multiples of $\mathrm{Id}_{\mathfrak{a}(V)}$.

Moreover, being a representation of $\widetilde{W}$, the map q yields a representation of the universal enveloping algebra $\mathsf{U}(\widetilde{W}) \equiv \mathfrak{q}(W)$. This representation is referred to as the *spinor-oscillator representation* of $\mathfrak{q}(W)$. In the sequel we will be interested in the $\mathfrak{osp}(W)$-representation induced from it by the isomorphism τ^{-1}.

There is a natural $\mathbb{Z}$-grading $\mathfrak{a}(V) = \bigoplus_{m \geq 0} \mathfrak{a}^m(V)$,

$$\mathfrak{a}^m(V) = \textstyle\bigoplus_{k+l=m} \wedge^k (V_1^*) \otimes \mathrm{S}^l(V_0^*) .$$

Note that the operators $\varepsilon(\varphi_1)$ and $\mu(\varphi_0)$ increase the $\mathbb{Z}$-degree of $\mathfrak{a}(V)$ by one, while the operators $\iota(v_1)$ and $\delta(v_0)$ decrease it by one. Note also if $\Lambda = (-\mathrm{Id}_V) \oplus \mathrm{Id}_{V^*}$ is the $\mathfrak{osp}$-element introduced in §2.2.2, then a direct computation using the formula (2.8) shows that $\mathfrak{a}^m(V)$ is an eigenspace of the operator $(q \circ \tau^{-1})(\Lambda)$ with eigenvalue m. Thus $\Lambda \in \mathfrak{osp}$ is represented on the spinor-oscillator module $\mathfrak{a}(V)$ by the *degree*.

2.6.1. Weight constraints. — We now specialize to the situation of $V = U \otimes \mathbb{C}^N$ with $U = U_0 \oplus U_1$ a $\mathbb{Z}_2$-graded vector space as in §2.3, and we require U_0 and U_1 to be isomorphic with dimension $\dim U_0 = \dim U_1 = n$. Recall that

$$(\mathfrak{osp}(U \oplus U^*), \mathfrak{o}_N) , \quad (\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*), \mathfrak{sp}_N) ,$$

are Howe dual pairs in $\mathfrak{osp}(W)$ which we denote by $(\mathfrak{g}, \mathfrak{k})$. There is a decomposition

$$\mathfrak{g} = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(2)} , \quad \mathfrak{g}^{(0)} = \mathfrak{g} \cap (\mathrm{End}(U) \oplus \mathrm{End}(U^*)) ,$$
$$\mathfrak{g}^{(-2)} = \mathfrak{g} \cap \mathrm{Hom}(U^*, U) , \quad \mathfrak{g}^{(2)} = \mathfrak{g} \cap \mathrm{Hom}(U, U^*) ,$$

in both cases. The notation highlights the fact that the operators in $\mathfrak{g}^{(m)} \hookrightarrow \mathfrak{osp}(V \oplus V^*)$ change the degree of elements in $\mathfrak{a}(V)$ by m. Note that the Cartan subalgebra $\mathfrak{h}$ of diagonal operators in $\mathfrak{g}$ is contained in $\mathfrak{g}^{(0)}$ but $\mathfrak{h} \neq \mathfrak{g}^{(0)}$.

Since the Lie algebra $\mathfrak{k}$ is defined on $\mathbb{C}^N$, the $\mathfrak{k}$-action on $\mathfrak{a}(V)$ preserves the degree. This action exponentiates to an action of the complex Lie group K on $\mathfrak{a}(V)$.

Proposition 2.2. — *The subalgebra $\mathfrak{a}(V)^K$ of K-invariants in $\mathfrak{a}(V)$ is an irreducible module for $\mathfrak{g}$. The vacuum $1 \in \mathfrak{a}(V)^K$ is contained in it as a cyclic vector such that*

$$\mathfrak{g}^{(-2)}.1 = 0 \, , \quad \mathfrak{g}^{(0)}.1 = \langle 1 \rangle_{\mathbb{C}} \, , \quad \langle \mathfrak{g}^{(2)}.1 \rangle_{\mathbb{C}} = \mathfrak{a}(V)^K \, .$$

Proof. — This is a restatement of Theorems 8 and 9 of [**7**]. □

Remark 2.2. — In the case of $(\mathfrak{g},\mathfrak{k}) = (\mathfrak{osp}(U \oplus U^*), \mathfrak{o}_N)$ it matters that $K = \mathrm{O}_N$, as the connected Lie group $K = \mathrm{SO}_N$ has invariants in $\mathfrak{a}(V)$ not contained in $\langle \mathfrak{g}^{(2)}.1 \rangle_{\mathbb{C}}$.

Proposition 2.2 has immediate consequences for the weights of the $\mathfrak{g}$-representation on $\mathfrak{a}(V)^K$. Using the notation of §2.2.1, let $\{H_{sj}\}$ be a standard basis of $\mathfrak{h}$ and $\{\vartheta_{sj}\}$ the corresponding dual basis. We now write $\vartheta_{0j} =: \phi_j$ and $\vartheta_{1j} =: \mathrm{i}\psi_j$ ($j = 1, \ldots, n$).

Corollary 2.3. — *The representations of $\mathfrak{osp}(U \oplus U^*)$ on $\mathfrak{a}(V)^{\mathrm{O}_N}$ and $\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)$ on $\mathfrak{a}(V)^{\mathrm{Sp}_N}$ each have highest weight $\lambda_N = \frac{N}{2}\sum_{j=1}^n (\mathrm{i}\psi_j - \phi_j)$. Every weight of these representations is of the form $\sum_{j=1}^n (\mathrm{i}m_j\psi_j - n_j\phi_j)$ with $-\frac{N}{2} \leq m_j \leq \frac{N}{2} \leq n_j$.*

Proof. — Recall from §2.3 the embedding of $\mathfrak{osp}(U \oplus U^*)$ and $\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)$ in $\mathfrak{osp}(W)$, and from Lemma 2.15 the isomorphism $\tau^{-1} : \mathfrak{osp}(W) \to \mathfrak{s}$ where $\mathfrak{s}$ is the Lie superalgebra of supersymmetrized degree-two elements in $\mathfrak{q}(W)$. Specializing formula (2.8) to the case of a Cartan algebra generator $H_{sj} \in \mathfrak{h} \subset \mathfrak{g}$ one gets

$$\tau^{-1}(H_{sj}) = -\tfrac{1}{2}\textstyle\sum_{a=1}^N \big((f_{s,j} \otimes f_a)(e_{s,j} \otimes e_a) + (-1)^s (e_{s,j} \otimes e_a)(f_{s,j} \otimes f_a)\big) \, ,$$

where $\{e_a\}$ is a basis of $\mathbb{C}^N$ and $\{f_a\}$ the dual basis of $(\mathbb{C}^N)^*$.

Now let $\tau^{-1}(H_{sj}) \in \mathfrak{s}$ act by the corresponding operator, say $\hat{H}_{sj} := (q \circ \tau^{-1})(H_{sj})$, in the spinor-oscillator representation q of $\mathfrak{s} \subset \mathfrak{q}(W)$. Application of that operator to the highest-weight vector $1 \in \mathbb{C} \equiv \wedge^0(V_1^*) \otimes \mathrm{S}^0(V_0^*) \subset \mathfrak{a}(V)^K$ yields

$$\hat{H}_{1j}\, 1 = \tfrac{1}{2}\textstyle\sum_a \iota(e_{1,j} \otimes e_a)\varepsilon(f_{1,j} \otimes f_a) 1 = \frac{N}{2} \, ,$$
$$\hat{H}_{0j}\, 1 = -\tfrac{1}{2}\textstyle\sum_a \delta(e_{0,j} \otimes e_a)\mu(f_{0,j} \otimes f_a) 1 = -\frac{N}{2} \, .$$

Altogether this means that $\hat{H}\, 1 = \lambda_N(H) 1$ where $\lambda_N(H) = \frac{N}{2}\sum_j (\mathrm{i}\psi_j(H) - \phi_j(H))$.

From Lemma 2.7 the roots α corresponding to root spaces $\mathfrak{g}_\alpha \subset \mathfrak{g}^{(2)}$ are of the form

$$-\phi_j - \phi_{j'} \, , \quad -\mathrm{i}\psi_j - \mathrm{i}\psi_{j'} \, , \quad -\phi_j - \mathrm{i}\psi_{j'} \, ,$$

where the indices j, j' are subject to restrictions that depend on $\mathfrak{g}$ being $\mathfrak{osp}(U \oplus U^*)$ or $\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)$. From $\mathfrak{a}(V)^K = \mathfrak{g}^{(2)}.1$ one then has $m_j \leq \frac{N}{2} \leq n_j$ for every weight $\gamma = \sum(\mathrm{i}m_j\psi_j - n_j\phi_j)$ of the $\mathfrak{g}$-representation on $\mathfrak{a}(V)^K$.

The restriction $m_j \geq \frac{N}{2} - N$ results from $\wedge(V_1^*) = \wedge(U_1^* \otimes (\mathbb{C}^N)^*)$ being isomorphic to $\otimes_{j=1}^n \wedge (\mathbb{C}^N)^*$ and the vanishing of $\wedge^k(\mathbb{C}^N)^* = 0$ for $k > N$. □

Corollary 2.4. — *For each of our two cases $\mathfrak{g} = \mathfrak{osp}(U \oplus U^*)$ and $\mathfrak{g} = \mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)$ the element $\Lambda = -\sum_{s,j} H_{sj} \subset \mathfrak{g}$ is represented on $\mathfrak{a}(V)^K$ by the degree operator.*

Proof. — Since the K-action on $\mathfrak{a}(V)$ preserves the degree, the subalgebra $\mathfrak{a}(V)^K$ is still $\mathbb{Z}$-graded by the same degree. Summing the above expressions for $(q \circ \tau^{-1})(H_{sj})$ over s, j and using CAR and CCR to combine terms, we obtain

$$(q \circ \tau^{-1})(\Lambda) = \sum_{j=1}^{n} \sum_{a=1}^{N} \big(\mu(f_{0,j} \otimes f_a) \delta(e_{0,j} \otimes e_a) + \varepsilon(f_{1,j} \otimes f_a) \iota(e_{1,j} \otimes e_a) \big) ,$$

which is in fact the operator for the degree of the $\mathbb{Z}$-graded module $\mathfrak{a}(V)^K$. □

2.6.2. Positive and simple roots. — We here record the systems of simple positive roots that we will use later (in §4.3.5). In the case of $\mathfrak{osp}(U \oplus U^*)$ this will be

$$\phi_1 - \phi_2 , \ldots , \phi_{n-1} - \phi_n , \phi_n - \mathrm{i}\psi_1 , \mathrm{i}\psi_1 - \mathrm{i}\psi_2 , \ldots , \mathrm{i}\psi_{n-1} - \mathrm{i}\psi_n , \mathrm{i}\psi_{n-1} + \mathrm{i}\psi_n .$$

The corresponding system of positive roots for $\mathfrak{osp}(U \oplus U^*)$ is

$$\phi_j \pm \phi_k , \mathrm{i}\psi_j \pm \mathrm{i}\psi_k \ (j < k) , \ 2\phi_j , \ \phi_j \pm \mathrm{i}\psi_k \ (j,k = 1, \ldots, n) .$$

In the case of $\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)$ we choose the system of simple positive roots

$$\phi_1 - \phi_2 , \ldots , \phi_{n-1} - \phi_n , \phi_n - \mathrm{i}\psi_1 , \mathrm{i}\psi_1 - \mathrm{i}\psi_2 , \ldots , \mathrm{i}\psi_{n-1} - \mathrm{i}\psi_n , 2\mathrm{i}\psi_n .$$

The corresponding positive root system then is

$$\phi_j \pm \phi_k , \mathrm{i}\psi_j \pm \mathrm{i}\psi_k \ (j < k) , \ 2\mathrm{i}\psi_j , \ \phi_j \pm \mathrm{i}\psi_k \ (j,k = 1, \ldots, n) .$$

In both cases the roots

$$\phi_j - \phi_k , \mathrm{i}\psi_j - \mathrm{i}\psi_k \ (j < k) , \quad \phi_j - \mathrm{i}\psi_k \ (j,k = 1, \ldots, n) ,$$

form a system of positive roots for $\mathfrak{gl}(U) \simeq \mathfrak{g}^{(0)} \subset \mathfrak{osp}$.

2.6.3. Unitary structure. — We now equip the spinor-oscillator module $\mathfrak{a}(V)$ for $V = V_0 \oplus V_1$ with a unitary structure. The idea is to think of the algebra $\mathfrak{a}(V)$ as a subset of $\mathscr{O}(V_0, \wedge V_1^*)$, the holomorphic functions $V_0 \to \wedge(V_1^*)$. For such functions a Hermitian scalar product is defined via Berezin's notion of superintegration as follows.

For present purposes, it is imperative that V be defined over $\mathbb{R}$, i.e., $V = V_\mathbb{R} \otimes \mathbb{C}$, and that V be re-interpreted as a *real* vector space $V' := V_\mathbb{R} \oplus J V_\mathbb{R}$ with complex structure $J \simeq \mathrm{i}$. Needless to say, this is done in a manner consistent with the $\mathbb{Z}_2$-grading, so that $V' = V_0' \oplus V_1'$ and $V_s' = V_{s,\mathbb{R}} \oplus J V_{s,\mathbb{R}} \simeq V_{s,\mathbb{R}} \otimes \mathbb{C} = V_s$.

From $V_s = U_s \otimes \mathbb{C}^N$ and $U_1 \simeq U_0$ we are given an isomorphism $V_1 \simeq V_0$. This induces a canonical isomorphism $\wedge(V_1'^*) \simeq \wedge(V_0'^*)$, which gives rise to a bundle isomorphism Ω sending $\Gamma(V_0', \wedge V_1'^*)$, the algebra of real-analytic functions on V_0' with values in $\wedge(V_1'^*)$, to $\Gamma(V_0', \wedge T^* V_0')$, the complex of real-analytic differential forms on V_0'. Fixing

some orientation of V_0', the Berezin (super-)integral for the $\mathbb{Z}_2$-graded vector space $V' = V_0' \oplus V_1'$ is then defined as the composite map

$$\Gamma(V_0', \wedge V_1'^*) \xrightarrow{\Omega} \Gamma(V_0', \wedge T^* V_0') \xrightarrow{\int} \mathbb{C}\,, \quad \Phi \mapsto \Omega[\Phi] \mapsto \int_{V_0'} \Omega[\Phi]\,,$$

whenever the integral over V_0' exists. Thus the Berezin integral is a two-step process: first the section Φ is converted into a differential form, then the form $\Omega[\Phi]$ is integrated in the usual sense to produce a complex number. Of course, by the rules of integration of differential forms only the top-degree component of $\Omega[\Phi]$ contributes to the integral.

The subspace $V_\mathbb{R} \subset V'$ has played no role so far, but now we use it to decompose the complexification $V' \otimes \mathbb{C}$ into holomorphic and anti-holomorphic parts: $V' \otimes \mathbb{C} = V \oplus \overline{V}$ and determine an operation of complex conjugation $V^* \to \overline{V^*}$. We also fix on $V = V_0 \oplus V_1$ a Hermitian scalar product (a.k.a. unitary structure) $\langle\, ,\, \rangle$ so that $V_0 \perp V_1$. This scalar product determines a parity-preserving complex anti-linear bijection $c : V \to V^*$ by $v \mapsto cv = \langle v, \rangle$. Composing c with complex conjugation $V^* \to \overline{V^*}$ we get a $\mathbb{C}$-linear isomorphism $V \to \overline{V^*}$, $v \mapsto \overline{cv}$.

In this setting there is a distinguished Gaussian section $\gamma \in \Gamma(V_0', \wedge V_1'^* \otimes \mathbb{C})$ singled out by the conditions

$$\forall v_0 \in V_0\,, v_1 \in V_1 : \quad \delta(v_0)\gamma = -\mu(\overline{cv_0})\gamma\,, \quad \iota(v_1)\gamma = -\varepsilon(\overline{cv_1})\gamma\,. \tag{2.10}$$

To get a close-up view of γ, let $\{e_{0,j}\}$ and $\{e_{1,j}\}$ be orthonormal bases of V_0 resp. V_1, and let $z_j = ce_{0,j}$ and $\zeta_j = ce_{1,j}$ be the corresponding coordinate functions, with complex conjugates $\bar{z}_j$ and $\overline{\zeta}_j$. Viewing ζ_j, $\overline{\zeta}_j$ as generators of $\wedge(V_1'^* \otimes \mathbb{C})$, our section $\gamma \in \Gamma(V_0', \wedge V_1'^* \otimes \mathbb{C})$ is the standard Gaussian

$$\gamma = \text{const} \times \mathrm{e}^{-\sum_j (z_j \bar{z}_j + \zeta_j \overline{\zeta}_j)}\,.$$

We fix the normalization of γ by the condition $\int_{V_0'} \Omega[\gamma] = 1$.

A unitary structure on the spinor-oscillator module $\mathfrak{a}(V)$ is now defined as follows. Let complex conjugation $V^* \to \overline{V^*}$ be extended to an algebra anti-homomorphism $\mathfrak{a}(V) \to \mathfrak{a}(\overline{V})$ by the convention $\overline{\Phi_1 \Phi_2} = \overline{\Phi}_2 \overline{\Phi}_1$ (without any sign factors). Then, if Φ_1, Φ_2 are any two elements of $\mathfrak{a}(V)$, we view them as holomorphic maps $V_0 \to \wedge(V_1^*)$, multiply $\overline{\Phi}_1$ with Φ_2 to form $\overline{\Phi}_1 \Phi_2 \in \Gamma(V_0', \wedge V_1'^* \otimes \mathbb{C})$, and define their Hermitian scalar product by

$$\langle \Phi_1, \Phi_2 \rangle_{\mathfrak{a}(V)} := \int_{V_0'} \Omega[\gamma \overline{\Phi}_1 \Phi_2]\,. \tag{2.11}$$

Let us mention in passing that (2.11) coincides with the unitary structure of $\mathfrak{a}(V)$ used in the Hamiltonian formulation of quantum field theories and in the Fock space description of many-particle systems composed of fermions and bosons. The elements

$$\bigwedge\nolimits_j \zeta_j^{m_j} \otimes \prod\nolimits_j z_j^{n_j} / \sqrt{n_j!} \tag{2.12}$$

for $m_j \in \{0, 1\}$ and $n_j \in \{0, 1, \ldots\}$ form an orthonormal set in $\mathfrak{a}(V)$, which in physics is called the occupation number basis of $\mathfrak{a}(V)$.

Lemma 2.16. — *For all $v_0 \in V_0$ and $v_1 \in V_1$ the pairs of operators $\delta(v_0)$, $\mu(cv_0)$ and $\iota(v_1)$, $\varepsilon(cv_1)$ in* $\mathrm{End}(\mathfrak{a}(V))$ *obey the relations*

$$\delta(v_0)^\dagger = \mu(cv_0) , \quad \iota(v_1)^\dagger = \varepsilon(cv_1) ,$$

i.e., they are mutual adjoints with respect to the unitary structure of $\mathfrak{a}(V)$.

Proof. — Let $v \in V_0$. Since $\overline{\Phi}_1 \in \mathfrak{a}(\overline{V})$ is anti-holomorphic, we have $\delta(v)\overline{\Phi}_1 = 0$. By the first defining property of γ in (2.10) and the fact that $\delta(v)$ is a derivation,

$$\gamma\overline{\Phi}_1\delta(v)\Phi_2 = \delta(v)\left(\gamma\overline{\Phi}_1\Phi_2\right) + \mu(\overline{cv})\gamma\overline{\Phi}_1\Phi_2 ,$$

and passing to the Hermitian scalar product by the Berezin integral we obtain

$$\langle\Phi_1, \delta(v)\Phi_2\rangle_{\mathfrak{a}(V)} = \int_{V_0} \Omega[\gamma\overline{\Phi}_1\overline{\mu(cv)}\Phi_2] = \langle\mu(cv)\Phi_1, \Phi_2\rangle_{\mathfrak{a}(V)} .$$

By the definition of the †-operation this means that $\delta(v)^\dagger = \mu(cv)$.

In the case of $v \in V_1$ the argument is similar but for a few sign changes. Our starting relation changes to

$$\gamma\overline{\Phi}_1\iota(v)\Phi_2 = (-1)^{|\Phi_1|}\iota(v)\left(\gamma\overline{\Phi}_1\Phi_2\right) + (-1)^{|\Phi_1|}\varepsilon(\overline{cv})\gamma\overline{\Phi}_1\Phi_2 ,$$

since the operator $\iota(v)$ is an *anti*-derivation. If $v \mapsto \tilde{v}$ denotes the isomorphism $V_1 \to V_0$, then $\Omega \circ \iota(v) = \iota(\tilde{v}) \circ \Omega$ and the first term on the right-hand side Berezin-integrates to zero because $\iota(\tilde{v})$ lowers the degree in $\wedge T^*V_0'$. Therefore,

$$\langle\Phi_1, \iota(v)\Phi_2\rangle_{\mathfrak{a}(V)} = \int_{V_0'} \Omega[\gamma\overline{\Phi}_1\overline{\varepsilon(cv)}\Phi_2] = \langle\varepsilon(cv)\Phi_1, \Phi_2\rangle_{\mathfrak{a}(V)} ,$$

which is the statement $\iota(v)^\dagger = \varepsilon(cv)$. □

By the Hermitian scalar product (2.11) and the corresponding L^2-norm, the spinor-oscillator module $\mathfrak{a}(V)$ is completed to a Hilbert space, $\mathscr{A}_V$. A nice feature here is that, as an immediate consequence of the factors $1/\sqrt{n_j\,!}$ in the orthonormal basis (2.12), the L^2-condition $\langle\Phi, \Phi\rangle_{\mathfrak{a}(V)} < \infty$ implies absolute convergence of the power series for $\Phi \in \mathscr{A}_V$. Hence $\mathscr{A}_V$ can be viewed as a subspace of $\mathscr{O}(V_0, \wedge V_1^*)$:

$$\mathscr{A}_V = \{\Phi \in \mathscr{O}(V_0, \wedge V_1^*) \mid \langle\Phi, \Phi\rangle_{\mathfrak{a}(V)} < \infty\} .$$

In the important case of isomorphic components $V_0 \simeq V_1$, we may regard $\mathscr{A}_V$ as the Hilbert space of square-integrable holomorphic differential forms on V_0.

Note that although $\delta(v)$ and $\mu(\varphi)$ do not exist as operators on the Hilbert space $\mathscr{A}_V$, they do extend to linear operators on $\mathscr{O}(V_0, \wedge V_1^*)$ for all $v \in V_0$ and $\varphi \in V_0^*$.

2.7. Real structures. — In this subsection we define a real structure for the complex vector space $W = V \oplus V^*$ and describe, in particular, the resulting real forms of the ($\mathbb{Z}_2$-even components of the) Howe dual partners introduced above.

Recalling the map $c : V \to V^*$, $v \mapsto \langle v, \rangle$, let $W_\mathbb{R} \simeq V$ be the vector space

$$W_\mathbb{R} = \{v + cv \mid v \in V\} \subset V \oplus V^* = W .$$

Note that $W_\mathbb{R}$ can be viewed as the fixed point set $W_\mathbb{R} = \mathrm{Fix}(C)$ of the involution

$$C : W \to W , \quad v + \varphi \mapsto c^{-1}\varphi + cv .$$

By the orthogonality assumption, $W_\mathbb{R} = W_{0,\mathbb{R}} \oplus W_{1,\mathbb{R}}$ where $W_{s,\mathbb{R}} = W_s \cap W_\mathbb{R}$.

The symmetric bilinear form S on $W_1 = V_1 \oplus V_1^*$ restricts to a Euclidean structure

$$S : W_{1,\mathbb{R}} \times W_{1,\mathbb{R}} \to \mathbb{R} , \quad (v + cv, v' + cv') \mapsto 2\,\mathfrak{Re}\langle v, v'\rangle ,$$

whereas the alternating form A on $W_0 = V_0 \oplus V_0^*$ induces a real-valued symplectic form

$$\omega = \mathrm{i}A : W_{0,\mathbb{R}} \times W_{0,\mathbb{R}} \to \mathbb{R} , \quad (v + cv, v' + cv') \mapsto 2\,\mathfrak{Im}\langle v, v'\rangle .$$

Please be warned that, since $Q = S + A$ fails to be real-valued on $W_\mathbb{R}$, the intersection $\mathfrak{osp}(W) \cap \mathrm{End}(W_\mathbb{R})$ is *not* a real form of the complex Lie superalgebra $\mathfrak{osp}(W)$.

The connected classical real Lie groups associated to the bilinear forms S and ω are

$$\mathrm{SO}(W_{1,\mathbb{R}}) := \{g \in \mathrm{SL}(W_{1,\mathbb{R}}) \mid \forall w, w' \in W_{1,\mathbb{R}} : S(gw, gw') = S(w, w')\} ,$$
$$\mathrm{Sp}(W_{0,\mathbb{R}}) := \{g \in \mathrm{GL}(W_{0,\mathbb{R}}) \mid \forall w, w' \in W_{0,\mathbb{R}} : \omega(gw, gw') = \omega(w, w')\} .$$

They have Lie algebras denoted by $\mathfrak{o}(W_{1,\mathbb{R}})$ and $\mathfrak{sp}(W_{0,\mathbb{R}})$. By construction we have $\mathfrak{osp}(W)_0 \cap \mathrm{End}(W_\mathbb{R}) \simeq \mathfrak{o}(W_{1,\mathbb{R}}) \oplus \mathfrak{sp}(W_{0,\mathbb{R}})$, and this in fact is a real form of the complex Lie algebra $\mathfrak{osp}(W)_0 \simeq \mathfrak{o}(W_1) \oplus \mathfrak{sp}(W_0)$.

Proposition 2.3. — *The elements of* $\mathfrak{o}(W_{1,\mathbb{R}}) \oplus \mathfrak{sp}(W_{0,\mathbb{R}}) \subset \mathfrak{osp}(W)$ *are mapped via* τ^{-1} *and the spinor-oscillator representation to anti-Hermitian operators in* $\mathrm{End}(\mathfrak{a}(V))$.

Proof. — Let $X \in \mathfrak{o}(W_{1,\mathbb{R}}) \oplus \mathfrak{sp}(W_{0,\mathbb{R}})$. We know from Lemma 2.15 that $\tau^{-1}(X)$ is a super-symmetrized element of degree two in the Clifford-Weyl algebra $\mathfrak{q}(W)$. To see the explicit form of such an element, recall the definition $\tau(a)w = [a, w]$. Since $Q = S + A$, and A restricts to $\mathrm{i}\omega$, the fundamental bracket $[\,,\,] : W_\mathbb{R} \times W_\mathbb{R} \to \mathbb{C}$ given by $[w, w'] = Q(w, w')$ is real-valued on $W_{1,\mathbb{R}}$ but imaginary-valued on $W_{0,\mathbb{R}}$. Therefore,

$$\tau^{-1}(\mathfrak{o}(W_{1,\mathbb{R}})) = \mathrm{span}_\mathbb{R}\{ww' - w'w\} \quad (w, w' \in W_{1,\mathbb{R}}) ,$$
$$\tau^{-1}(\mathfrak{sp}(W_{0,\mathbb{R}})) = \mathrm{span}_\mathbb{R}\{\mathrm{i}ww' + \mathrm{i}w'w\} \quad (w, w' \in W_{0,\mathbb{R}}) .$$

The proposed statement $X^\dagger = -X$ now follows under the assumption that the spinor-oscillator representation maps every $w \in W_\mathbb{R}$ to a self-adjoint operator in $\mathrm{End}(\mathfrak{a}(V))$. But every element $w \in W_\mathbb{R}$ is of the form $v_1 + cv_1 + v_0 + cv_0$ and this maps to the operator $\iota(v_1) + \varepsilon(cv_1) + \delta(v_0) + \mu(cv_0)$, which is self-adjoint by Lemma 2.16. □

Given the real structure $W_\mathbb{R}$ of W, we now ask how $\mathrm{End}(W_\mathbb{R})$ intersects with the Howe pairs $(\mathfrak{osp}(U \oplus U^*), \mathfrak{o}_N)$ and $(\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*), \mathfrak{sp}_N)$ embedded in $\mathfrak{osp}(W)$. By the observation that Q restricted to $W_\mathbb{R}$ is not real-valued, $\mathfrak{osp}(U \oplus U^*) \cap \mathrm{End}(W_\mathbb{R})$ fails to be a real form of the complex Lie superalgebra $\mathfrak{osp}(U \oplus U^*)$, and the same goes for $\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)$. Nevertheless, it is still true that the even components of these intersections are real forms of the complex Lie algebras $\mathfrak{osp}(U \oplus U^*)_0$ and $\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)_0$.

The real forms of interest are best understood by expressing them via blocks with respect to the decomposition $W = V \oplus V^*$. Since $W_{\mathbb{R}} = \mathrm{Fix}(C)$, the complex linear endomorphisms of W stabilizing $W_{\mathbb{R}}$ are given by

$$\mathrm{End}(W_{\mathbb{R}}) \simeq \{X \in \mathrm{End}(W) \mid X = CXC^{-1}\} .$$

Writing X in block-decomposed form

$$X = \mathsf{A} \oplus \mathsf{B} \oplus \mathsf{C} \oplus \mathsf{D} \equiv \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} ,$$

where $\mathsf{A} \in \mathrm{End}(V)$, $\mathsf{B} \in \mathrm{Hom}(V^*, V)$, $\mathsf{C} \in \mathrm{Hom}(V, V^*)$, and $\mathsf{D} \in \mathrm{End}(V^*)$, the condition $X = CXC^{-1}$ becomes

$$\mathsf{C} = \overline{\mathsf{B}} , \quad \mathsf{D} = \overline{\mathsf{A}} .$$

The bar here is a short-hand notation for the complex anti-linear maps

$$\mathrm{Hom}(V^*, V) \to \mathrm{Hom}(V, V^*) , \quad \mathsf{B} \mapsto \overline{\mathsf{B}} := c\mathsf{B}c ,$$
$$\mathrm{End}(V) \to \mathrm{End}(V^*) , \quad \mathsf{A} \mapsto \overline{\mathsf{A}} := c\mathsf{A}c^{-1} .$$

When expressed with respect to compatible bases of V and V^*, these maps are just the standard operation of taking the complex conjugate of the matrices of A and B.

Now, to get an understanding of the intersections $\mathfrak{o}_N \cap \mathrm{End}(W_{\mathbb{R}})$ and $\mathfrak{sp}_N \cap \mathrm{End}(W_{\mathbb{R}})$, recall the relation $\mathsf{D} = -\mathsf{A}^{\mathrm{t}}$ for $X \in \mathfrak{osp}(W)_0$ and the fact that the action of the complex Lie algebras $\mathfrak{o}_N = \mathfrak{o}(\mathbb{C}^N)$ and $\mathfrak{sp}_N = \mathfrak{sp}(\mathbb{C}^N)$ on W stabilizes the decomposition $W = V \oplus V^*$, with the implication that $\mathsf{B} = \mathsf{C} = 0$ in both cases. Combining $\mathsf{D} = -\mathsf{A}^{\mathrm{t}}$ with $\mathsf{D} = \overline{\mathsf{A}}$ one gets the anti-Hermitian property $\mathsf{A} = -\overline{\mathsf{A}}^{\mathrm{t}}$, which means that $\mathfrak{o}_N \cap \mathrm{End}(W_{\mathbb{R}})$ and $\mathfrak{sp}_N \cap \mathrm{End}(W_{\mathbb{R}})$ are compact real forms of $\mathfrak{o}_N$ and $\mathfrak{sp}_N$.

Turning to the Howe dual partners of $\mathfrak{o}_N$ and $\mathfrak{sp}_N$, recall from §2.3 the isomorphism $\psi : \mathbb{C}^N \to (\mathbb{C}^N)^*$ and arrange for it to be an isometry, $\overline{\psi^{-1}} = \psi^{\mathrm{t}}$, of the unitary structures of $\mathbb{C}^N$ and $(\mathbb{C}^N)^*$. Recall also the embedding of the two Lie superalgebras $\mathfrak{osp}(U \oplus U^*)$ and $\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)$ into $\mathfrak{osp}(W) = \mathfrak{osp}(U \otimes \mathbb{C}^N \oplus U^* \otimes (\mathbb{C}^N)^*)$ by

$$\begin{pmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{pmatrix} \mapsto \begin{pmatrix} \mathsf{a} \otimes \mathrm{Id} & \mathsf{b} \otimes \psi^{-1} \\ \mathsf{c} \otimes \psi & \mathsf{d} \otimes \mathrm{Id} \end{pmatrix} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} .$$

Here the notation still means the same, i.e., $\mathsf{a} \in \mathrm{End}(U)$, $\mathsf{b} \in \mathrm{Hom}(U^*, U)$, and so on.

Let a real structure $(U \oplus U^*)_{\mathbb{R}}$ of $U \oplus U^*$ be defined in the same way as the real structure $W_{\mathbb{R}} = (V \oplus V^*)_{\mathbb{R}}$ of $W = V \oplus V^*$.

Proposition 2.4. — $\mathfrak{osp}(U \oplus U^*)_0 \cap \mathrm{End}(W_{\mathbb{R}}) \simeq \mathfrak{o}((U_1 \oplus U_1^*)_{\mathbb{R}}) \oplus \mathfrak{sp}((U_0 \oplus U_0^*)_{\mathbb{R}})$.

Proof. — The intersection is computed by transferring the conditions $\mathsf{D} = \overline{\mathsf{A}}$ and $\mathsf{C} = \overline{\mathsf{B}}$ to the level of $\mathfrak{osp}(U \oplus U^*)_0$. Of course $\mathsf{D} = \overline{\mathsf{A}}$ just reduces to the corresponding condition $\mathsf{d} = \overline{\mathsf{a}}$. Because the isometry $\psi : \mathbb{C}^N \to (\mathbb{C}^N)^*$ in the present case is symmetric one has $\overline{\psi^{-1}} = \psi^{\mathrm{t}} = +\psi$, so the condition $\mathsf{C} = \overline{\mathsf{B}}$ transfers to $\mathsf{c} = \overline{\mathsf{b}}$. For the same reason, the parity of the maps b, c is identical to that of B, C, i.e., $\mathsf{b}|_{U_0^* \to U_0}$ is symmetric, $\mathsf{b}|_{U_1^* \to U_1}$ is skew, and similar for c. Hence, computing the intersection

$\mathfrak{osp}(U\oplus U^*)_0\cap\mathrm{End}(W_\mathbb{R})$ amounts to the same as computing $\mathfrak{osp}(V\oplus V^*)_0\cap\mathrm{End}(W_\mathbb{R})$, and the statement follows from our previous discussion of the latter case. ☐

In the case of the Howe pair $(\mathfrak{osp}(\widetilde{U}\oplus\widetilde{U}^*),\mathfrak{sp}_N)$ the isometry $\psi:\mathbb{C}^N\to(\mathbb{C}^N)^*$ is skew, so that $\overline{\psi^{-1}}=\psi^{\mathrm{t}}=-\psi$. At the same time, the parity of b,c is reversed as compared to B,C: now the map $\mathsf{b}|_{U_0^*\to U_0}$ is skew and $\mathsf{b}|_{U_1^*\to U_1}$ is symmetric (and similar for c). Therefore,

$$\mathfrak{osp}(\widetilde{U}\oplus\widetilde{U}^*)_0\cap\mathrm{End}(W_{1,\mathbb{R}})\simeq\{\begin{pmatrix}\mathsf{a}&\mathsf{b}\\-\overline{\mathsf{b}}&-\mathsf{a}^{\mathrm{t}}\end{pmatrix}\in\mathrm{End}(U_1\oplus U_1^*)\mid\mathsf{a}=-\overline{\mathsf{a}}^{\mathrm{t}}\,,\,\mathsf{b}=+\mathsf{b}^{\mathrm{t}}\}\,,$$

which is a compact real form $\mathfrak{usp}(U_1\oplus U_1^*)$ of $\mathfrak{sp}(U_1\oplus U_1^*)$; and

$$\mathfrak{osp}(\widetilde{U}\oplus\widetilde{U}^*)_0\cap\mathrm{End}(W_{0,\mathbb{R}})\simeq\{\begin{pmatrix}\mathsf{a}&\mathsf{b}\\-\overline{\mathsf{b}}&-\mathsf{a}^{\mathrm{t}}\end{pmatrix}\in\mathrm{End}(U_0\oplus U_0^*)\mid\mathsf{a}=-\overline{\mathsf{a}}^{\mathrm{t}}\,,\,\mathsf{b}=-\mathsf{b}^{\mathrm{t}}\}\,,$$

which is a non-compact real form of $\mathfrak{o}(U_0\oplus U_0^*)$ known as $\mathfrak{so}^*(U_0\oplus U_0^*)$.

Let us summarize this result.

Proposition 2.5. — $\mathfrak{osp}(\widetilde{U}\oplus\widetilde{U}^*)_0\cap\mathrm{End}(W_\mathbb{R})\simeq\mathfrak{usp}(U_1\oplus U_1^*)\oplus\mathfrak{so}^*(U_0\oplus U_0^*)$.

3. Semigroup representation

As before, we identify the complex Lie superalgebra $\mathfrak{g}:=\mathfrak{osp}(W)$ with the space of super-symmetrized degree-two elements in $\mathfrak{q}_2(W)$, so that

$$\mathfrak{q}_2(W)=\mathfrak{g}\oplus\mathfrak{q}_1(W)\,,\quad\mathfrak{q}_1(W)=W\oplus\mathbb{C}\,.$$

The adjoint representation of $\mathfrak{g}$ on $\mathfrak{q}(W)$ restricts to the Lie algebra representation of $\mathfrak{g}_0=\mathfrak{o}(W_1)\oplus\mathfrak{sp}(W_0)$ on $W=W_1\oplus W_0$ which is just the direct sum of the fundamental representations of $\mathfrak{o}(W_1)$ and $\mathfrak{sp}(W_0)$. These are integrated by the fundamental representations of the complex Lie groups $\mathrm{SO}(W_1)$ and $\mathrm{Sp}(W_0)$ respectively.

Since the Clifford-Weyl algebra $\mathfrak{q}(W)$ is an associative algebra, one can ask if, given $x\in\mathfrak{g}_0\subset\mathfrak{q}(W)$, the exponential series e^x makes sense. The existence of a one-parameter group e^{tx} for $x\in\mathfrak{g}_0$ would of course imply that

$$\frac{d}{dt}\,\mathrm{e}^{tx}\,w\,\mathrm{e}^{-tx}\Big|_{t=0}=\mathrm{ad}(x)\,w\qquad(w\in W)\,.\tag{3.1}$$

Now $\mathfrak{q}(W)=\mathfrak{c}(W_1)\otimes\mathfrak{w}(W_0)$. Since the Clifford algebra $\mathfrak{c}(W_1)$ is finite-dimensional, the series e^x for $x\in\mathfrak{o}(W_1)\hookrightarrow\mathfrak{g}_0$ does make immediate sense. In this way one is able to exponentiate the Lie algebra $\mathfrak{o}(W_1)$ in $\mathfrak{c}(W_1)$. The associated complex Lie group, which is then embedded in $\mathfrak{c}(W_1)$, is $\mathrm{Spin}(W_1)$. This is a 2:1 cover of the complex orthogonal group $\mathrm{SO}(W_1)$. Its conjugation representation on W_1 as in (3.1) realizes the covering map as a homomorphism $\mathrm{Spin}(W_1)\to\mathrm{SO}(W_1)$.

Viewing the other summand $\mathfrak{sp}(W_0)$ of $\mathfrak{g}_0$ as being in the infinite-dimensional Weyl algebra $\mathfrak{w}(W_0)$, it is definitely not possibly to exponentiate it in such a naive way. This is in particular due to the fact that for most $x\in\mathfrak{sp}(W_0)$ the formal series e^x is not contained in any space $\mathfrak{w}_n(W_0)$ of the filtration of $\mathfrak{w}(W_0)$.

As a first step toward remedying this situation, we consider $\mathfrak{q}(W)$ as a space of densely defined operators on the completion $\mathscr{A}_V$ (cf. §2.6.3) of the spinor-oscillator module $\mathfrak{a}(V)$. Since all difficulties are on the W_0 side, for the remainder of this chapter we simplify the notation by letting $W := W_0$ and discussing only the oscillator representation of $\mathfrak{w}(W)$. Recall that this representation on $\mathfrak{a}(V)$ is defined by multiplication $\mu(\varphi)$ for $\varphi \in V^*$ and the directional derivative $\delta(v)$ for $v \in V$.

For $x \in \mathfrak{w}(W)$ there is at least no formal obstruction to the exponential series of x existing in $\mathrm{End}(\mathscr{A}_V)$. However, direct inspection shows that convergence cannot be expected unless some restrictions are imposed on x. This is done by introducing a notion of unitarity and an associated semigroup of contraction operators.

3.1. The oscillator semigroup. — Here we introduce the basic semigroup in the complex symplectic group. Various structures are lifted to its canonical 2:1 covering. Actions of the real symplectic and metaplectic groups are discussed along with the role played by the cone of elliptic elements.

3.1.1. Contraction semigroup: definitions, basic properties. — Letting $\langle \, , \rangle$ be the unitary structure on V which was fixed in the previous chapter, we recall the complex anti-linear bijection $c : V \to V^*$, $v \mapsto \langle v , \rangle$. There is an induced map $C : W \to W$ on $W = V \oplus V^*$ by $C(v + \varphi) = c^{-1}\varphi + cv$. As before, we put $W_\mathbb{R} := \mathrm{Fix}(C) \subset W$.

Since we have restricted our attention to the symplectic side, the vector spaces W and $W_\mathbb{R}$ are now equipped with the standard complex symplectic structure A and real symplectic form $\omega = \mathrm{i}A$ respectively. From here on in this chapter we abbreviate the notation by writing $\mathrm{Sp} := \mathrm{Sp}(W)$ and $\mathfrak{sp} := \mathfrak{sp}(W)$. Let an anti-unitary involution $\sigma : \mathrm{Sp} \to \mathrm{Sp}$ be defined by $g \mapsto CgC^{-1}$. Its fixed point group $\mathrm{Fix}(\sigma)$ is the real form $\mathrm{Sp}(W_\mathbb{R})$ of main interest. We here denote it by $\mathrm{Sp}_\mathbb{R}$ and let $\mathfrak{sp}_\mathbb{R}$ stand for its Lie algebra.

Given A and C, consider the mixed-signature Hermitian structure

$$W \times W \to \mathbb{C} , \quad (w, w') \mapsto A(Cw, w') ,$$

which we denote by $A(Cw, w') =: \langle w, w' \rangle_s$, with subscript s to distinguish it from the canonical Hermitian structure of W given by $\langle v + \varphi, v' + \varphi' \rangle := \langle v, v' \rangle + \langle c^{-1}\varphi', c^{-1}\varphi \rangle$. The relation between the two is

$$\langle w, w' \rangle_s = \langle w, sw' \rangle , \qquad s = (-\mathrm{Id}_V) \oplus \mathrm{Id}_{V^*} .$$

Note also the relation

$$\sigma(g) = CgC^{-1} = s\,(g^{-1})^\dagger s \qquad (g \in \mathrm{Sp}) .$$

Now observe that the real form $\mathrm{Sp}_\mathbb{R}$ is the subgroup of $\langle \, , \rangle_s$-isometries in Sp :

$$\mathrm{Sp}_\mathbb{R} = \{ g \in \mathrm{Sp} \mid \forall w \in W : \ \langle gw, gw \rangle_s = \langle w, w \rangle_s \} .$$

Then define a semigroup $\mathrm{H}(W^s)$ in Sp by

$$\mathrm{H}(W^s) := \{ h \in \mathrm{Sp} \mid \forall w \in W, \ w \neq 0 : \ \langle hw, hw \rangle_s < \langle w, w \rangle_s \} .$$

Note that the operation $g \mapsto g^\dagger$ of Hermitian conjugation with respect to $\langle \, , \rangle$ stabilizes Sp and that $\mathrm{Sp}_\mathbb{R}$ is defined by the condition $g^\dagger s g = s$. The semigroup $\mathrm{H}(W^s)$ is defined

by $h^\dagger sh < s$, or equivalently, $s - h^\dagger sh$ is positive definite. We will see later that $\mathrm{H}(W^s)$ (or, rather, a $2:1$ cover thereof) acts by contraction operators on the Hilbert space $\mathscr{A}_V$.

It is immediate that $\mathrm{H}(W^s)$ is an open semigroup in Sp with $\mathrm{Sp}_\mathbb{R}$ on its boundary. Furthermore, $\mathrm{H}(W^s)$ is stabilized by the action of $\mathrm{Sp}_\mathbb{R} \times \mathrm{Sp}_\mathbb{R}$ by $h \mapsto g_1 h g_2^{-1}$.

The map $\pi : \mathrm{Sp} \to \mathrm{Sp}$, $h \mapsto h\sigma(h^{-1})$, will play an important role in our considerations. It is invariant under the $\mathrm{Sp}_\mathbb{R}$-action by right multiplication, $\pi(hg^{-1}) = \pi(h)$, and is equivariant with respect to the action defined by left multiplication on its domain of definition and conjugation on its image space, $\pi(gh) = g\pi(h)g^{-1}$. Direct calculation shows that in fact the π-fibers are exactly the orbits of the $\mathrm{Sp}_\mathbb{R}$-action by right multiplication. Observe that if $h = \exp(\mathrm{i}X)$ for $X \in \mathfrak{sp}_\mathbb{R}$, then $\sigma(h) = h^{-1}$ and $\pi(h) = h^2$. In particular, if $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{sp}$ which is defined over $\mathbb{R}$, then $\pi|_{\exp(\mathrm{i}\mathfrak{t}_\mathbb{R})}$ is just the squaring map $t \mapsto t^2$.

3.1.2. Actions of $\mathrm{Sp}_\mathbb{R}$. — We now fix a Cartan subalgebra $\mathfrak{t}$ having the property that $T_\mathbb{R} = \exp(\mathfrak{t}_\mathbb{R})$ is contained in the unitary maximal compact subgroup defined by $\langle\, , \rangle$ of $\mathrm{Sp}(W_\mathbb{R})$. This means that T acts diagonally on the decomposition $W = V \oplus V^*$ and there is a (unique up to order) orthogonal decomposition $V = E_1 \oplus \ldots \oplus E_d$ into one-dimensional subspaces so that if $F_j := c(E_j)$, then T acts via characters χ_j on the vector spaces $P_j := E_j \oplus F_j$ by $t(e_j, f_j) = (\chi_j(t)e_j, \chi_j(t)^{-1} f_j)$.

In other words, we may choose $\{e_j\}_{j=1,\ldots,d}$ to be an orthonormal basis of V and equip V^* with the dual basis so that the elements $t \in T$ are in diagonal form:

$$t = \mathrm{diag}(\lambda_1, \ldots, \lambda_d, \lambda_1^{-1}, \ldots, \lambda_d^{-1}) , \qquad \lambda_j = \chi_j(t) .$$

Observe that, conversely, the elements of Sp that stabilize the decomposition $W = P_1 \oplus \ldots \oplus P_d$ and act diagonally in the above sense, are exactly the elements of T. Moreover, $\exp(\mathrm{i}\mathfrak{t}_\mathbb{R})$ is the subgroup of elements $t \in T$ with $\chi_j(t) \in \mathbb{R}_+$ for all j. Note that the complex symplectic planes P_j are A-orthogonal and defined over $\mathbb{R}$.

We now wish to analyze $\mathrm{H}(W^s)$ via the map $\pi : h \mapsto h\sigma(h^{-1})$. However, for a technical reason related to the proof of Proposition 3.1 below, we must begin with the opposite map, $\pi' : h \mapsto \sigma(h^{-1})h$. Thus let $M := \pi'(\mathrm{H}(W^s))$ and write $\pi' : \mathrm{H}(W^s) \to M$. The toral semigroup $T_+ := \exp(\mathrm{i}\mathfrak{t}_\mathbb{R}) \cap \mathrm{H}(W^s)$ consists of those elements $t \in \exp(\mathrm{i}\mathfrak{t}_\mathbb{R})$ that act as contractions on V^*, i.e., $0 < \chi_j(t)^{-1} < 1$ for all j. The restriction $\pi'|_{T_+} = \pi|_{T_+}$ is, as indicated above, the squaring map $t \mapsto t^2$; in particular we have $T_+ \subset M$ and the set $\{gtg^{-1} \mid t \in T_+ , g \in \mathrm{Sp}_\mathbb{R}\}$ is likewise contained in M.

In the sequel, we will often encounter the action of $\mathrm{Sp}_\mathbb{R}$ on T_+ and M by conjugation. We therefore denote this action by a special name, $\mathrm{Int}(g)\,t := gtg^{-1}$.

Proposition 3.1. — $M = \mathrm{Int}(\mathrm{Sp}_\mathbb{R})T_+$.

Proof. — For $g \in \mathrm{Sp}$ one has $\sigma(g^{-1}) = Cg^{-1}C^{-1} = sg^\dagger s$. Hence if $M \ni m = \sigma(h^{-1})h$ with $h \in \mathrm{H}(W^s)$, then $m = sh^\dagger sh$. Consequently, $\langle w, mw\rangle_s = \langle hw, hw\rangle_s < \langle w, w\rangle_s$ for all $w \in W \setminus \{0\}$. In particular, $\langle w, mw\rangle_s \in \mathbb{R}$ and if $w \neq 0$ is an m-eigenvector with eigenvalue λ, it follows that $\lambda \in \mathbb{R}$ and $\lambda\langle w, w\rangle_s < \langle w, w\rangle_s \neq 0$.

Now we have $ChC^{-1} = \sigma(h)$ and hence $CmC^{-1} = m^{-1}$. As a result, if $w \neq 0$ is an m-eigenvector with eigenvalue λ, then so is Cw with eigenvalue λ^{-1}. Since $CsC^{-1} = -s$,

the product of $\langle w,w\rangle_s$ with $\langle Cw,Cw\rangle_s$ is always negative. If $\langle w,w\rangle_s > 0$ it follows that $\lambda < 1$ and $\lambda^{-1} > 1$; if $\langle Cw,Cw\rangle_s > 0$ then $\lambda^{-1} < 1$ and $\lambda > 1$. In both cases $0 < \lambda \neq 1$.

Since m does indeed have at least one eigenvector, we have constructed a complex 2-plane Q_1 as the span of the linearly independent vectors w and Cw. The plane Q_1 is defined over $\mathbb{R}$ and, because $0 \neq \langle w,w\rangle_s = A(Cw,w)$, it is A-nondegenerate. Its A-orthogonal complement $Q_1^\perp$ is therefore also nondegenerate and defined over $\mathbb{R}$.

The transformation $m \in \mathrm{Sp}$ stabilizes the decomposition $W = Q_1 \oplus Q_1^\perp$. Hence, proceeding by induction we obtain an A-orthogonal decomposition $W = Q_1 \oplus \ldots \oplus Q_d$. Since the Q_j are m-invariant symplectic planes defined over $\mathbb{R}$, there exists $g \in \mathrm{Sp}_\mathbb{R}$ so that $t := gmg^{-1}$ stabilizes the above T-invariant decomposition $W = P_1 \oplus \ldots \oplus P_d$. Exchanging w with Cw if necessary, we may assume that t acts diagonally on $P_j = E_j \oplus F_j$ by $(e_j, f_j) \mapsto (\lambda_j e_j, \lambda_j^{-1} f_j)$ with $\lambda_j > 1$. In other words, $t \in T_+$. ☐

If we let

$$\mathrm{Sp}_\mathbb{R} T_+ \mathrm{Sp}_\mathbb{R} := \{ g_1 t g_2^{-1} \mid g_1, g_2 \in \mathrm{Sp}_\mathbb{R}, t \in T_+ \},$$

then we now have the following analog of the KAK-decomposition.

Corollary 3.1. — *The semigroup* $\mathrm{H}(W^s)$ *decomposes as* $\mathrm{H}(W^s) = \mathrm{Sp}_\mathbb{R} T_+ \mathrm{Sp}_\mathbb{R}$. *In particular,* $\mathrm{H}(W^s)$ *is connected.*

Proof. — By definition, $\mathrm{H}(W^s) = \pi'^{-1}(M)$, and from Proposition 3.1 one has $\mathrm{H}(W^s) = \pi'^{-1}(\mathrm{Int}(\mathrm{Sp}_\mathbb{R})T_+)$. Now the map $\pi'|_{T_+} : T_+ \to T_+$, $t \mapsto t^2$ is surjective. Therefore $\pi'^{-1}(T_+) = \mathrm{Sp}_\mathbb{R} T_+$, which is to say that each point in the fiber of π' over $t \in T_+$ lies in the orbit of $\sqrt{t} \in T_+$ generated by left multiplication with $\mathrm{Sp}_\mathbb{R}$. On the other hand, by the property $g\pi'(h)g^{-1} = \pi'(hg^{-1})$ of $\mathrm{Sp}_\mathbb{R}$-equivariance we have

$$\mathrm{Int}(\mathrm{Sp}_\mathbb{R})T_+ = \mathrm{Int}(\mathrm{Sp}_\mathbb{R})\pi'(\pi'^{-1}(T_+)) = \pi'(\pi'^{-1}(T_+)\mathrm{Sp}_\mathbb{R})$$

and hence $\mathrm{H}(W^s) = \pi'^{-1}(\mathrm{Int}(\mathrm{Sp}_\mathbb{R})T_+) = \pi'^{-1}(T_+)\mathrm{Sp}_\mathbb{R} = \mathrm{Sp}_\mathbb{R} T_+ \mathrm{Sp}_\mathbb{R}$.

Because $\mathrm{Sp}_\mathbb{R}$ and T_+ are connected, so is $\mathrm{H}(W^s) = \mathrm{Sp}_\mathbb{R} T_+ \mathrm{Sp}_\mathbb{R}$. ☐

It is clear that $M = \mathrm{Int}(\mathrm{Sp}_\mathbb{R})T_+ \subset \mathrm{H}(W^s)$. Furthermore, since both T_+ and $\mathrm{Sp}_\mathbb{R}$ are invariant under the operation of Hermitian conjugation $h \mapsto h^\dagger$ and under the involution $h \mapsto shs$, we have the following consequences.

Corollary 3.2. — $\mathrm{H}(W^s)$ *is invariant under* $h \mapsto h^\dagger$ *and also under* $h \mapsto shs$. *In particular,* $\mathrm{H}(W^s)$ *is stabilized by the map* $h \mapsto \sigma(h^{-1}) = sh^\dagger s$.

Remark 3.1. — Letting $h' := \sigma(h)^{-1}$ one has $\pi'(h) = \sigma(h)^{-1}h = h'\sigma(h')^{-1} = \pi(h')$ and hence $M = \pi'(\mathrm{H}(W^s)) = \pi(\mathrm{H}(W^s))$. The stability of $\mathrm{H}(W^s)$ under $h \mapsto \sigma(h)^{-1}$ was not immediate from our definition of $\mathrm{H}(W^s)$, which is why we have been working from the viewpoint of $\mathrm{H}(W^s) = \pi'^{-1}(M)$ so far. Now that we have it, we may regard $\mathrm{H}(W^s)$ as the total space of an $\mathrm{Sp}_\mathbb{R}$-principal bundle $\pi : \mathrm{H}(W^s) \to M$. We are going to see in Corollary 3.3 that this principal bundle is trivial.

Next observe that, since $\sigma(m) = m^{-1}$ for $m = \sigma(h)^{-1}h = h'\sigma(h')^{-1} \in M$, the maps $\pi : M \to M$ and $\pi' : M \to M$ coincide and are just the square $m \mapsto m^2$. Thus the claim that the elements of M have a unique square root in M can be formulated as follows.

Proposition 3.2. — *The squaring map $\pi = \pi' : M \to M$ is bijective.*

Proof. — Recall from Proposition 3.1 that every $m \in M$ is diagonalizable in the sense that $M = \mathrm{Int}(\mathrm{Sp}_{\mathbb{R}})T_+$. Since $\pi : T_+ \to T_+$ is surjective, the surjectivity of $\pi : M \to M$ is immediate. For the injectivity of π we note that the m-eigenspace with eigenvalue λ is contained in the m^2-eigenspace with eigenvalue λ^2. The result then follows from the fact that all eigenvalues of m are positive real numbers. □

Corollary 3.3. — *Each of the two $\mathrm{Sp}_{\mathbb{R}}$-equivariant maps $\mathrm{Sp}_{\mathbb{R}} \times M \to \mathrm{H}(W^s)$ defined by $(g,m) \mapsto gm$ and by $(g,m) \mapsto mg^{-1}$, is a bijection.*

Proof. — Consider the map $(g,m) \mapsto gm$. Surjectivity is evident from Corollary 3.1 and $M = \mathrm{Int}(\mathrm{Sp}_{\mathbb{R}})T_+$. For the injectivity it suffices to prove that if $m_1, m_2 \in M$ and $g \in \mathrm{Sp}_{\mathbb{R}}$ with $gm_1 = m_2$, then $m_1 = m_2$. But this follows directly from $\pi'(m_1) = \pi'(gm_1) = \pi'(m_2)$ and the fact that $\pi'|_M$ is the bijective squaring map.

The proof for the map $(g,m) \mapsto mg^{-1}$ is similar, with π replacing π'. □

3.1.3. Cone realization of M. — Let us look more carefully at M as a geometric object. First, as we have seen, the elements m of M satisfy the condition $m = \sigma(m^{-1})$. We regard $\psi : \mathrm{H}(W^s) \to \mathrm{H}(W^s)$, $h \mapsto \sigma(h^{-1})$, as an anti-holomorphic involution and reformulate this condition as $M \subset \mathrm{Fix}(\psi)$. In the present section we are going to show that M is a closed, connected, real-analytic submanifold of $\mathrm{H}(W^s)$ which locally agrees with $\mathrm{Fix}(\psi)$. This implies in particular that M is totally real in $\mathrm{H}(W^s)$ with $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} \mathrm{H}(W^s)$. We will also show that the exponential map identifies M with a precisely defined open cone in $i\mathfrak{sp}_{\mathbb{R}}$. We begin with the following statement.

Lemma 3.1. — *The image M of π is closed as a subset of $\mathrm{H}(W^s)$.*

Proof. — Let $h \in \mathrm{cl}(M) \subset \mathrm{H}(W^s)$. By the definition of M, we still have $h\sigma(h)^{-1} =: m \in M$. If h_n is any sequence from M with $h_n \to h$, then $h_n\sigma(h_n)^{-1} \to m$. But m has a unique square root $\sqrt{m} \in M$ and $h_n = \sigma(h_n)^{-1} \to \sqrt{m}$. Hence $h = \sqrt{m} \in M$. □

Remark 3.2. — M of course fails to be closed as a subset of Sp. For example, $g = \mathrm{Id}$ is in the closure of $M \subset \mathrm{Sp}$ but is not in M.

Lemma 3.2. — *The exponential map $\exp : \mathfrak{sp} \to \mathrm{Sp}$ has maximal rank along $\mathfrak{t}_+$.*

Proof. — We are going to use the fact that the squaring map $S : \mathrm{Sp} \to \mathrm{Sp}$, $g \mapsto g^2$, is a local diffeomorphism of Sp at any point $t \in T_+$. To show this, we compute the differential of S at t and obtain

$$D_t S = dL_{t^2} \circ (\mathrm{Id}_{\mathfrak{sp}} + \mathrm{Ad}(t^{-1})) \circ dL_{t^{-1}} ,$$

where dL_g denotes the differential of the left translation $L_g : \mathrm{Sp} \to \mathrm{Sp}$, $g_1 \mapsto gg_1$. The middle map $\mathrm{Id}_{\mathfrak{sp}} + \mathrm{Ad}(t^{-1}) : \mathfrak{sp} \to \mathfrak{sp}$ is regular because all of the eigenvalues of

$t \in T_+$ are positive real numbers. Since $dL_{t^{-1}} : T_t\mathrm{Sp} \to \mathfrak{sp}$ and $dL_{t^2} : \mathfrak{sp} \to T_{t^2}\mathrm{Sp}$ are isomorphisms, it follows that $D_tS : T_t\mathrm{Sp} \to T_{t^2}\mathrm{Sp}$ is an isomorphism.

Turning to the proof of the lemma, given $\xi \in \mathfrak{t}_+$ we now choose $n \in \mathbb{N}$ so that $2^{-n}\xi$ is in a neighborhood of $0 \in \mathfrak{sp}$ where exp is a diffeomorphism. It follows that for U a sufficiently small neighborhood of ξ, the exponential map expressed as

$$U \ni \eta \mapsto 2^{-n}\eta \mapsto \exp(2^{-n}\eta) \overset{S^n}{\mapsto} (\exp(2^{-n}\eta))^{2^n} = \exp(\eta)$$

is a diffeomorphism of U onto its image. $\square$

Now recall $M = \mathrm{Int}(\mathrm{Sp}_\mathbb{R})T_+$ and consider the cone

$$\mathscr{C} := \mathrm{Ad}(\mathrm{Sp}_\mathbb{R})\mathfrak{t}_+ \subset i\mathfrak{sp}_\mathbb{R}\ .$$

It follows from the equivariance of exp, i.e., $\exp(\mathrm{Ad}(g)\xi) = \mathrm{Int}(g)\exp(\xi)$, that $\exp : \mathscr{C} \to \mathrm{Int}(\mathrm{Sp}_\mathbb{R})T_+ = M$ is surjective. Furthermore, $\exp|_{\mathfrak{t}_+} : \mathfrak{t}_+ \to T_+$ is injective and for every $\xi \in \mathfrak{t}_+$ the isotropy groups of the $\mathrm{Sp}_\mathbb{R}$-actions at ξ and $\exp(\xi)$ are the same. Therefore $\exp : \mathscr{C} \to M$ is also injective.

In fact, much stronger regularity holds. For the statement of this result we recall the anti-holomorphic involution $\psi : \mathrm{H}(W^s) \to \mathrm{H}(W^s)$ defined by $h \mapsto \sigma(h^{-1})$ and let $\mathrm{Fix}(\psi)^0$ denote the connected component of $\mathrm{Fix}(\psi)$ that contains M.

Proposition 3.3. — *The image* $M \subset \mathrm{H}(W^s)$ *of* $\pi : h \mapsto h\sigma(h^{-1})$ *is the closed, connected, totally real submanifold* $\mathrm{Fix}(\psi)^0$*, which is half-dimensional in the sense that* $\dim_\mathbb{R} M = \dim_\mathbb{C} \mathrm{H}(W^s)$*. The set* $\mathscr{C} = \mathrm{Ad}(\mathrm{Sp}_\mathbb{R})\mathfrak{t}_+$*, which is an open positive cone in* $i\mathfrak{sp}_\mathbb{R}$*, is in bijection with M by the real-analytic diffeomorphism* $\exp : \mathscr{C} \to M$.

Proof. — Lemma 3.2 implies that $\mathfrak{t}_+$ possesses an open neighborhood U in $i\mathfrak{sp}_\mathbb{R}$ so that $\exp|_U$ is everywhere of maximal rank. Because T_+ lies in $\mathrm{H}(W^s)$ and $\mathrm{H}(W^s)$ is open in Sp, by choosing U small enough we may assume that $\exp(\frac{1}{2}U) \subset \mathrm{H}(W^s)$ and therefore that $\exp(U) \subset M = \exp(\mathscr{C})$. Since exp is a local diffeomorphism on U, we may also assume that $U \subset \mathscr{C} = \mathrm{Ad}(\mathrm{Sp}_\mathbb{R})\mathfrak{t}_+$, and it then follows that $\mathscr{C} = \mathrm{Ad}(\mathrm{Sp}_\mathbb{R})U$. In particular, this shows that $\mathscr{C}$ is open in $i\mathfrak{sp}_\mathbb{R}$. In summary,

$$\mathscr{C} = \mathrm{Ad}(\mathrm{Sp}_\mathbb{R})U \xrightarrow{\exp} \mathrm{Int}(\mathrm{Sp}_\mathbb{R})\exp(U) = M \subset \mathrm{Fix}(\psi)^0\ .$$

By the equivariance of exp, we also know that it is everywhere of maximal rank on $\mathscr{C}$.

Now ψ is an anti-holomorphic involution. Therefore, $\mathrm{Fix}(\psi)^0$ is a totally real, half-dimensional closed submanifold of $\mathrm{H}(W^s)$, and since $\mathscr{C}$ is open in $i\mathfrak{sp}_\mathbb{R}$, we also know that $\dim_\mathbb{C}\mathscr{C} = \dim_\mathbb{R}\mathrm{Fix}(\psi)^0$. The maximal rank property of exp then implies that $M = \mathrm{im}(\exp : \mathscr{C} \to \mathrm{Fix}(\psi)^0)$ is open in $\mathrm{Fix}(\psi)^0$. In Lemma 3.1 it was shown that M is closed in $\mathrm{H}(W^s)$. Thus it is open and closed in the connected manifold $\mathrm{Fix}(\psi)^0$, and consequently $\exp : \mathscr{C} \to M = \mathrm{Fix}(\psi)^0$ is a local diffeomorphism of manifolds. Since we already know that $\exp : \mathscr{C} \to M$ is bijective, the desired result follows. $\square$

Corollary 3.4. — *The two identifications* $\mathrm{Sp}_\mathbb{R} \times M = \mathrm{H}(W^s)$ *defined by* $(g,m) \mapsto gm$ *and* $(g,m) \mapsto mg^{-1}$ *are real-analytic diffeomorphisms. The fundamental group of* $\mathrm{H}(W^s)$ *is isomorphic to* $\pi_1(\mathrm{Sp}_\mathbb{R}) \simeq \mathbb{Z}$.

Proof. — The first statement is proved by explicitly constructing a smooth inverse to each of the two maps. For this let $g'm' = mg^{-1} = h \in \mathrm{H}(W^s)$ and note that $m = \sqrt{\pi(h)}$ and $m' = \sqrt{\pi'(h)}$. Since the square root is a smooth map on M, a smooth inverse in the two cases is defined by

$$h \mapsto (h\sqrt{\pi'(h)}^{\,-1}, \sqrt{\pi'(h)})\,, \quad \text{resp.} \quad h \mapsto (h^{-1}\sqrt{\pi(h)}, \sqrt{\pi(h)})\,.$$

The second statement follows from $\mathscr{C} \simeq M$ by exp and the fact that $\mathrm{Sp}_{\mathbb{R}}$ is a product of a cell and a maximal compact subgroup K. We choose K to be the unitary group $\mathrm{U} = \mathrm{U}(V, \langle\, ,\,\rangle)$ acting diagonally on $W = V \oplus V^*$ and recall that $\pi_1(\mathrm{U}) \simeq \mathbb{Z}$. ☐

3.2. Oscillator semigroup and metaplectic group. — Recall that we are concerned with the Lie algebra representation of $\mathfrak{sp}_{\mathbb{R}} \subset \mathfrak{sp}$ which is defined by the identification of $\mathfrak{sp}$ with the set of symmetrized elements of degree two in the Weyl algebra $\mathfrak{w}(W)$ and the representation of $\mathfrak{w}(W)$ on $\mathfrak{a}(V)$. In §3.4 we construct the oscillator representation of the metaplectic group Mp, a 2:1 cover of $\mathrm{Sp}_{\mathbb{R}}$, which integrates this Lie algebra representation. Observe that since $\pi_1(\mathrm{Sp}_{\mathbb{R}}) \simeq \mathbb{Z}$ and $\mathbb{Z}$ has only one subgroup of index two, there is a unique such covering $\tau : \mathrm{Mp} \to \mathrm{Sp}_{\mathbb{R}}$. The method of construction [**8**] we use first yields a representation of the 2:1 covering space $\widetilde{\mathrm{H}}(W^s)$ of $\mathrm{H}(W^s)$ and then realizes the oscillator representation of Mp by taking limits that correspond to going to $\mathrm{Sp}_{\mathbb{R}}$ in the boundary of $\mathrm{H}(W^s)$. This representation of the *oscillator semigroup* $\widetilde{\mathrm{H}}(W^s)$ is for our purposes at least as important as the representation of Mp.

The goal of the present section is to lift all essential structures on $\mathrm{H}(W^s)$ to $\widetilde{\mathrm{H}}(W^s)$.

3.2.1. Lifting the semigroup. — We begin by recalling a few basic facts about covering spaces. If G is a connected Lie group, its universal covering space U carries a canonical group structure: an element $u \in U$ in the fiber over $g \in G$ is a homotopy class $u \equiv [\alpha_g]$ of paths $\alpha_g : [0,1] \to G$ connecting g with the neutral element $e \in G$; and an associative product $U \times U \to U$, $(u_1, u_2) \mapsto u_1 u_2$, is defined by taking $u_1 u_2$ to be the unique homotopy class which is given by pointwise multiplication of any two paths representing the homotopy classes u_1, u_2. The fundamental group $\pi_1(G) \equiv \pi_1(G,e)$ acts on U by monodromy, i.e., if $[\alpha_g] = u \in U$ and $[c] = \gamma \in \pi_1(G)$, then one sets $\gamma(u) := [\alpha_g * c] \in U$ where $\alpha_g * c$ is the path from g to e which is obtained by composing the path α_g with the loop c based at e. This $\pi_1(G)$-action satisfies the compatibility condition $\gamma_1(u_1)\gamma_2(u_2) = (\gamma_1\gamma_2)(u_1u_2)$ and in that sense is central.

The situation for our semigroup $\mathrm{H}(W^s)$ is analogous except for the minor complication that the distinguished point $e = \mathrm{Id}$ does not lie in $\mathrm{H}(W^s)$ but lies in the closure of $\mathrm{H}(W^s)$. Hence, by the same principles, the universal cover U of $\mathrm{H}(W^s)$ comes with a product operation and there is a central action of $\pi_1(\mathrm{H}(W^s))$ on U. Moreover, the product $U \times U \to U$ still is associative. To see this, first notice that the subsemigroup $T_+ \subset \mathrm{H}(W^s)$ is simply connected and as such is canonically embedded in U. Then for $u_1, u_2, u_3 \in U$ observe that $u_1(u_2u_3) = \gamma((u_1u_2)u_3)$ where $\gamma \in \pi_1(\mathrm{H}(W^s))$ could theoretically depend on the u_j. However, any such dependence has to be continuous and the fundamental group is discrete, so in fact γ is independent of the u_j and, since γ is the identity when the u_j are in T_+ (lifted to U), the associativity follows.

Let now $\Gamma \simeq 2\mathbb{Z}$ denote the subgroup of index two in $\pi_1(\mathrm{H}(W^s)) \simeq \mathbb{Z}$ and consider $\widetilde{\mathrm{H}}(W^s) := U/\Gamma$, which is our object of interest. Since the Γ-action on U is central, i.e., $\gamma_1(u_1)\gamma_2(u_2) = (\gamma_1\gamma_2)(u_1u_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$ and $u_1, u_2 \in U$, the product $U \times U \to U$ descends to a product $U/\Gamma \times U/\Gamma \to U/\Gamma$. Thus $U/\Gamma = \widetilde{\mathrm{H}}(W^s)$ is a semigroup, and the situation at hand is summarized by the following statement.

Proposition 3.4. — *The 2:1 covering* $\tau_H : U/\Gamma = \widetilde{\mathrm{H}}(W^s) \to \mathrm{H}(W^s)$, $[\alpha_h]\,\Gamma \mapsto h$, *is a homomorphism of semigroups.*

3.2.2. Actions of the metaplectic group. — Recall that we have two 2:1 coverings: a homomorphism of groups $\tau : \mathrm{Mp} \to \mathrm{Sp}_{\mathbb{R}}$, along with a homomorphism of semigroups $\tau_H : \widetilde{\mathrm{H}}(W^s) \to \mathrm{H}(W^s)$. Now, a pair of elements $(g', g) \in \mathrm{Sp}_{\mathbb{R}} \times \mathrm{Sp}_{\mathbb{R}}$ determines a transformation $h \mapsto g'hg^{-1}$ of $\mathrm{H}(W^s)$, and by the homotopy lifting property of covering maps a corresponding action of $\mathrm{Mp} \times \mathrm{Mp}$ on $\widetilde{\mathrm{H}}(W^s)$ is obtained as follows.

Consider the canonical mapping $\mathrm{Mp} \times M \to \mathrm{Sp}_{\mathbb{R}} \times M$ given by τ. By the real-analytic diffeomorphism $\mathrm{Sp}_{\mathbb{R}} \times M \to \mathrm{H}(W^s)$, $(g, m) \mapsto gm$, this map can be regarded as a 2:1 covering of $\mathrm{H}(W^s)$, and since any two 2:1 coverings are isomorphic we get an identification of the covering space $\widetilde{\mathrm{H}}(W^s)$ with $\mathrm{Mp} \times M$. Moreover, the action of the group Mp on itself by left translations induces on $\mathrm{Mp} \times M \simeq \widetilde{\mathrm{H}}(W^s)$ an Mp-action which, by construction, satisfies the relation

$$\tau_H(g \cdot h) = \tau(g)\tau_H(h) \quad (g \in \mathrm{Mp},\ h \in \widetilde{\mathrm{H}}(W^s))\ .$$

This can be viewed as a statement of Mp-equivariance of the covering map τ_H.

Now, we have another real-analytic diffeomorphism $\mathrm{Sp}_{\mathbb{R}} \times M \to \mathrm{H}(W^s)$ by $(g, m) \mapsto mg^{-1}$, which transfers left translation in $\mathrm{Sp}_{\mathbb{R}}$ to right multiplication on $\mathrm{H}(W^s)$, and by using it we can repeat the above construction. The result is another identification $\widetilde{\mathrm{H}}(W^s) \simeq \mathrm{Mp} \times M$ and another Mp-action on $\widetilde{\mathrm{H}}(W^s)$. Altogether we then have two actions of Mp on $\widetilde{\mathrm{H}}(W^s)$. The essence of the next statement is that they commute.

Proposition 3.5. — *There is a real-analytic action* $(g_1, g_2) \mapsto (g_1, g_2) \cdot h$ *of* $\mathrm{Mp} \times \mathrm{Mp}$ *on* $\widetilde{\mathrm{H}}(W^s)$ *such that the covering* $\tau_H : \widetilde{\mathrm{H}}(W^s) \to \mathrm{H}(W^s)$ *is* $(\mathrm{Mp} \times \mathrm{Mp})$*-equivariant:*

$$\tau_H((g_1, g_2) \cdot h) = \tau(g_1)\tau_H(h)\tau(g_2)^{-1}\ .$$

Proof. — By construction, the stated equivariance property of τ_H holds for each of the two actions of Mp separately. It then follows that it holds for all $(g_1, g_2) \in \mathrm{Mp} \times \mathrm{Mp}$ if the two actions commute. But by $\tau_H((g_1, e) \cdot h) = \tau(g_1)\tau_H(h)$ and $\tau_H((e, g_2) \cdot h) = \tau_H(h)\tau(g_2)^{-1}$ the commutator

$$g := (g_1, e)(e, g_2)(g_1, e)^{-1}(e, g_2)^{-1}$$

acts trivially on $\mathrm{H}(W^s)$ by τ_H, i.e., $\tau_H(g \cdot h) = \tau_H(h)$. Therefore g can be regarded as being in the covering group $\tau^{-1}(\mathrm{Id}) = \mathbb{Z}_2$ of the covering $\tau : \mathrm{Mp} \to \mathrm{Sp}_{\mathbb{R}}$. Since we can connect both g_1 and g_2 to the identity $e \in \mathrm{Mp}$ by a continuous curve, it follows from the discreteness of $\mathbb{Z}_2$ that $g \in \mathrm{Mp} \times \mathrm{Mp}$ acts trivially on $\widetilde{\mathrm{H}}(W^s)$. □

Notice that since the submanifold $M \subset \mathrm{H}(W^s)$ is simply connected, there exists a canonical lifting of M (which we still denote by M) to the cover $\widetilde{\mathrm{H}}(W^s)$; this is the unique lifting by which $T_+ \subset M$ is embedded as a subsemigroup in $\widetilde{\mathrm{H}}(W^s)$. Proposition 3.5 then allows us to write $\widetilde{\mathrm{H}}(W^s) = \mathrm{Mp}.M.\mathrm{Mp}$.

3.2.3. Lifting involutions. — Let us now turn to the issue of lifting the various involutions at hand. As a first remark, we observe that any Lie group automorphism $\varphi : \mathrm{Sp}_\mathbb{R} \to \mathrm{Sp}_\mathbb{R}$ uniquely lifts to a Lie group automorphism $\widetilde{\varphi}$ of the universal covering group $\widetilde{\mathrm{Sp}}_\mathbb{R}$, and the latter induces an automorphism of the fundamental group $\pi_1(\mathrm{Sp}_\mathbb{R}) \simeq \mathbb{Z}$ viewed as a subgroup of the center of $\widetilde{\mathrm{Sp}}_\mathbb{R}$. Now $\mathrm{Aut}(\pi_1(\mathrm{Sp}_\mathbb{R})) \simeq \mathrm{Aut}(\mathbb{Z}) \simeq \mathbb{Z}_2$ and both elements of this automorphism group stabilize the subgroup $\Gamma \simeq 2\mathbb{Z}$ in $\pi_1(\mathrm{Sp}_\mathbb{R})$. Therefore $\widetilde{\varphi}$ induces an automorphism of $\mathrm{Mp} = \widetilde{\mathrm{Sp}}_\mathbb{R}/\Gamma$.

Since the operation $h \mapsto h^{-1}$ canonically lifts from $\mathrm{Sp}_\mathbb{R}$ to Mp and $h \mapsto (h^{-1})^\dagger$ is a Lie group automorphism of $\mathrm{Sp}_\mathbb{R}$, it follows that Hermitian conjugation $h \mapsto h^\dagger$ has a natural lift to Mp. The same goes for the Lie group automorphism $h \mapsto shs$ of $\mathrm{Sp}_\mathbb{R}$.

Proposition 3.6. — *Hermitian conjugation $h \mapsto h^\dagger$ and the involution $h \mapsto shs$ lift to unique maps with the property that they stabilize the lifted manifold M. In particular, the basic anti-holomorphic map ψ : $\mathrm{H}(W^s) \to \mathrm{H}(W^s)$, $h \mapsto \sigma(h^{-1}) = sh^\dagger s$ lifts to an anti-holomorphic map $\widetilde{\psi}$: $\widetilde{\mathrm{H}}(W^s) \to \widetilde{\mathrm{H}}(W^s)$ which is the identity on M and* $\mathrm{Mp} \times \mathrm{Mp}$*-equivariant in that $\widetilde{\psi}(g_1 x g_2^{-1}) = g_2 \widetilde{\psi}(x) g_1^{-1}$ for all $g_1, g_2 \in$* Mp *and $x \in \widetilde{\mathrm{H}}(W^s)$.*

Proof. — Recall that the simply connected space $M \subset \mathrm{H}(W^s)$ has a canonical lifting (still denoted by M) to $\widetilde{\mathrm{H}}(W^s)$. Since all of our involutions stabilize M as a submanifold of $\mathrm{H}(W^s)$, they are canonically defined on the lifted manifold M. In particular, the involution ψ on M is the identity map, and therefore so is the lifted involution $\widetilde{\psi}$.

Note furthermore that the involution defined by $h \mapsto shs$ is holomorphic on $\mathrm{H}(W^s)$ and that the other two are anti-holomorphic. Now $\widetilde{\mathrm{H}}(W^s)$ is connected and the lifted version of M is a totally real submanifold of $\widetilde{\mathrm{H}}(W^s)$ with $\dim_\mathbb{R} M = \dim_\mathbb{C} \widetilde{\mathrm{H}}(W^s)$. In such a situation the identity principle of complex analysis implies that there exists at most one extension (holomorphic or anti-holomorphic) of an involution from M to $\widetilde{\mathrm{H}}(W^s)$. Therefore, it is enough to prove the existence of extensions.

Since $h \in \widetilde{\mathrm{H}}(W^s)$ is uniquely representable as $h = gm$ with $g \in \mathrm{Mp}$ and $m \in M$, the involution $h \mapsto h^\dagger$ is extended by $gm \mapsto (gm)^\dagger = m^\dagger g^\dagger$. Similarly, $h \mapsto shs$ extends by $gm \mapsto (sgs)(sms)$, and $h \mapsto sh^\dagger s$ does so by the composition of the other two.

The equivariance property of $\widetilde{\psi}$ follows from the fact that $g \mapsto sg^\dagger s$ on Mp coincides with the operation of taking the inverse, $g \mapsto g^{-1}$. □

3.3. Oscillator semigroup representation. — Here we construct the fundamental representation of the semigroup $\widetilde{\mathrm{H}}(W^s)$ on the Hilbert space $\mathscr{A}_V$, which in the present context we call Fock space. Our approach is parallel to that of Howe [**8**]: the Fock space we use is related to the L^2-space of Howe's work by the Bargmann transform [**6**]. (Using the language of physics, one would say that Howe works with the position wave function while our treatment relies on the phase space wave function.) In particular,

following Howe we take advantage of a realization of $\mathrm{H}(W^s)$ as the complement of a certain determinantal variety in the Siegel upper half plane.

3.3.1. Cayley transformation. — Let us begin with some background information on the Cayley transformation, which is defined to be the meromorphic mapping

$$C:\ \mathrm{End}(W) \to \mathrm{End}(W), \quad g \mapsto \frac{\mathrm{Id}_W + g}{\mathrm{Id}_W - g}.$$

If $g \in \mathrm{Sp}$, then from $A(gw, gw') = A(w, w')$ we have

$$A((\mathrm{Id}_W + g)w, (\mathrm{Id}_W - g)w') + A((\mathrm{Id}_W - g)w, (\mathrm{Id}_W + g)w') = 0$$

for all $w, w' \in W$. By assuming that $(\mathrm{Id}_W - g)$ is invertible and then replacing w and w' by $(\mathrm{Id}_W - g)^{-1}w$ resp. $(\mathrm{Id}_W - g)^{-1}w'$, we see that C maps the complement of the determinantal variety $\{g \in \mathrm{Sp} \mid \mathrm{Det}(\mathrm{Id}_W - g) = 0\}$ into $\mathfrak{sp}$.

The inverse of the Cayley transformation is given by

$$g = C^{-1}(X) = \frac{X - \mathrm{Id}_W}{X + \mathrm{Id}_W}.$$

Reversing the above argument, one shows that if $(X + \mathrm{Id}_W)$ is invertible and $X \in \mathfrak{sp}$, then $C^{-1}(X) \in \mathrm{Sp}$. Moreover, by the relation $X + \mathrm{Id}_W = 2(\mathrm{Id}_W - g)^{-1}$ for $C(g) = X$, if $\mathrm{Id}_W - g$ is regular, then so is $X + \mathrm{Id}_W$, and vice versa. Thus if we introduce the sets

$$D_{\mathrm{Sp}} := \{g \in \mathrm{Sp} \mid \mathrm{Det}(\mathrm{Id}_W - g) = 0\}, \quad D_{\mathfrak{sp}} := \{X \in \mathfrak{sp} \mid \mathrm{Det}(X + \mathrm{Id}_W) = 0\},$$

the following is immediate.

Proposition 3.7. — *The Cayley transformation defines a bi-holomorphic map*

$$C:\ \mathrm{Sp} \setminus D_{\mathrm{Sp}} \to \mathfrak{sp} \setminus D_{\mathfrak{sp}}.$$

Now we consider the restriction of C to the semigroup $\mathrm{H}(W^s)$. Letting $\dagger$ be the Hermitian conjugation of the previous section, denote by $\mathfrak{Re}(X) = \frac{1}{2}(X + X^\dagger)$ the real part of an operator $X \in \mathrm{End}(W)$ and define the associated Siegel upper half space $\mathfrak{S}$ to be the subset of elements $X \in \mathrm{End}(W)$ which are symmetric with respect to the canonical symmetric bilinear form S on $W = V \oplus V^*$ with $\mathfrak{Re}(X) > 0$. Notice the relations $S(w, w') = A(w, sw')$ and $A(sw, sw') = -A(w, w')$, from which it is seen that X is symmetric if and only if $sX \in \mathfrak{sp}$. Define $D_{\mathfrak{S}} := \{X \in \mathfrak{S} \mid \mathrm{Det}(sX + \mathrm{Id}_W) = 0\}$, let

$$\zeta_s^+ := \mathfrak{S} \setminus D_{\mathfrak{S}},$$

and define a slightly modified Cayley transformation by

$$a(g) := s\,\frac{\mathrm{Id}_W + g}{\mathrm{Id}_W - g}.$$

Translating Proposition 3.7, it follows that a defines a bi-holomorphic map from $\mathrm{Sp} \setminus D_{\mathrm{Sp}}$ onto the set of S-symmetric operators with $D_{\mathfrak{S}}$ removed.

Proposition 3.8. — *The modified Cayley transformation* $a:\ \mathrm{Sp} \setminus D_{\mathrm{Sp}} \to \mathrm{End}(W)$ *given by* $g \mapsto s\,(\mathrm{Id}_W + g)(\mathrm{Id}_W - g)^{-1}$ *restricts to a bi-holomorphic map*

$$a:\ \mathrm{H}(W^s) \to \zeta_s^+.$$

This result is an immediate consequence of the following identity.

Lemma 3.3. — *For $g \in \mathrm{Sp} \setminus D_{\mathrm{Sp}}$ let $a(g) = X$ and define $Y := (sX + \mathrm{Id}_W)^{-1}$. Then*

$$\tfrac{1}{2}(s - g^\dagger s g) = Y^\dagger (X + X^\dagger) Y \,.$$

In particular, one has the following equivalence:

$$\big(\mathfrak{Re}(X) > 0 \text{ and } \mathrm{Det}(sX + \mathrm{Id}_W) \neq 0\big) \;\Leftrightarrow\; s - g^\dagger s g > 0 \,.$$

Proof. — It is convenient to rewrite $s - g^\dagger s g$ as

$$s - g^\dagger s g = \tfrac{1}{2}(\mathrm{Id}_W - g^\dagger) s (\mathrm{Id}_W + g) + \tfrac{1}{2}(\mathrm{Id}_W + g^\dagger) s (\mathrm{Id}_W - g) \,.$$

Using $a(g) = X$ one directly computes that

$$\tfrac{1}{2}(\mathrm{Id}_W - g) = (sX + \mathrm{Id}_W)^{-1} \quad \text{and} \quad \tfrac{1}{2}(\mathrm{Id}_W + g) = (sX + \mathrm{Id}_W)^{-1} sX \,.$$

The desired identity follows by inserting these relations in the previous equation. □

Remark 3.3. — The modified Cayley transformation intertwines the anti-holomorphic involution $\psi : h \mapsto s h^\dagger s$ with the operation of taking the Hermitian conjugate $X \mapsto X^\dagger$:

$$a(\psi(h)) = a(h)^\dagger \,.$$

Since ζ_s^+ is obviously stable under Hermitian conjugation, this is another proof of the stability of $\mathrm{H}(W^s)$ under the involution ψ; cf. Corollary 3.2.

3.3.2. Construction of the semigroup representation. — Let us now turn to the main goal of this section. Recall that we have a Lie algebra representation of $\mathfrak{sp}$ on $\mathfrak{a}(V) = \mathrm{S}(V^*)$ which is defined by its canonical embedding in $\mathfrak{w}_2(W)$. We shall now construct the corresponding representation of the semigroup $\widetilde{\mathrm{H}}(W^s)$ on the Fock space $\mathscr{A}_V$.

It will be seen later that the character of this representation on the lifted toral semigroup T_+ is $\mathrm{Det}^{-\frac{1}{2}}(s - sh)$. This extends to $M = \mathrm{Int}(\mathrm{Mp})T_+$ by the invariance of the character with respect to the conjugation action of Mp. Since $\widetilde{\mathrm{H}}(W^s)$ is connected and M is totally real of maximal dimension in $\widetilde{\mathrm{H}}(W^s)$, the identity principle then implies that if a semigroup representation of $\widetilde{\mathrm{H}}(W^s)$ can be constructed with a holomorphic character, this character must be given by the square root function $h \mapsto \mathrm{Det}^{-\frac{1}{2}}(s - sh)$.

There is no difficulty discussing the square root on the simply connected submanifold M. However, in order to make sense of the square root function on the full semigroup, we must lift all considerations to $\widetilde{\mathrm{H}}(W^s)$. For convenience of notation, given $h \in \mathrm{H}(W^s)$ we let $a_h := a(h)$, and for $x \in \widetilde{\mathrm{H}}(W^s)$ we simply write $a_x \equiv a(\tau_H(x))$ where $\tau_H : \widetilde{\mathrm{H}}(W^s) \to \mathrm{H}(W^s)$ is the canonical covering map. Then we put

$$f(h) := \mathrm{Det}(a_h + s) = \mathrm{Det}(2s(\mathrm{Id}_W - h)^{-1}) \,,$$

and wish to define $\phi : \widetilde{\mathrm{H}}(W^s) \to \mathbb{C}$ to be the square root of f which agrees with the positive square root on T_+. (Here we regard T_+ as being in $\widetilde{\mathrm{H}}(W^s)$ by its canonical lifting as a subsemigroup of $\mathrm{H}(W^s)$ as in the previous section). This is possible because ϕ is naturally defined on $\{(\xi, \eta) \in \mathrm{H}(W^s) \times \mathbb{C} \mid f(\xi) = \eta^2\}$ which is itself a 2:1 cover of $\mathrm{H}(W^s)$. Since up to equivariant equivalence there is only one such covering,

namely $\widetilde{\mathrm{H}}(W^s) \to \mathrm{H}(W^s)$, it follows that we may define ϕ on $\widetilde{\mathrm{H}}(W^s)$ as desired. For the construction of the oscillator representation it is useful to observe that ϕ can be extended to a slightly larger space. This extension is constructed as follows.

Regard the complex symplectic group Sp as the total space of an $\mathrm{Sp}_{\mathbb{R}}$-principal bundle $\pi : \mathrm{Sp} \to \pi(\mathrm{Sp})$, $g \mapsto g\sigma(g^{-1})$. Recall that the restricted map $\pi : M \to M$ is a diffeomorphism, and that M contains the neutral element $\mathrm{Id} \in \mathrm{Sp}$ in its boundary. We choose a small ball B centered at Id in the base $\pi(\mathrm{Sp})$, and using the fact that M can be identified with a cone in $i\mathfrak{sp}_{\mathbb{R}}$ we observe that $A := B \cup M \subset \pi(\mathrm{Sp})$ is contractible. Now $U := \pi^{-1}(A)$ is diffeomorphic to a product $\mathrm{Sp}_{\mathbb{R}} \times A$ and thus comes with a 2:1 covering $\widetilde{U} \to U$ defined by $\tau : \mathrm{Mp} \to \mathrm{Sp}_{\mathbb{R}}$. The covering space $\widetilde{U}$ contains $\widetilde{\mathrm{H}}(W^s)$, and is invariant under the Mp-action by right multiplication. By construction it also contains the metaplectic group Mp, which covers the group $\mathrm{Sp}_{\mathbb{R}}$ in Sp.

Recall the definition of the determinant variety $D_{\mathrm{Sp}} = \{g \in \mathrm{Sp} \mid \mathrm{Det}(\mathrm{Id}_W - g) = 0\}$. Let $\widetilde{D}_{\mathrm{Sp}}$ denote the set of points in $\widetilde{U}$ which lie over $D_{\mathrm{Sp}} \cap U$ by the covering $\widetilde{U} \to U$.

Proposition 3.9. — *There is a unique continuous extension of ϕ from $\widetilde{\mathrm{H}}(W^s)$ to its closure in $\widetilde{U}$ so that ϕ^2 agrees with the lift of f from U. The intersection of $\widetilde{D}_{\mathrm{Sp}}$ with any* Mp-*orbit in $\widetilde{U}$ is nowhere dense in that orbit and the restriction of the extended function ϕ to the complement of that intersection is real-analytic.*

Remark 3.4. — Before beginning the proof, it should be clarified that at the points of the lifted determinant variety, i.e., where the lifted square root ϕ of the function $f(g) = \mathrm{Det}(2s\,(\mathrm{Id}_W - g)^{-1})$ has a pole, *continuity of the extension* means that the reciprocal of ϕ extends to a continuous function which vanishes on that set.

Proof. — The intersection of D_{Sp} with any $\mathrm{Sp}_{\mathbb{R}}$-orbit in U is nowhere dense in that orbit; therefore the same holds for the intersection of $\widetilde{D}_{\mathrm{Sp}}$ with any Mp-orbit in $\widetilde{U}$.

Let $x \in \widetilde{U} \setminus \widetilde{D}_{\mathrm{Sp}}$ be a point of the boundary of $\widetilde{\mathrm{H}}(W^s)$. Choose a local contractible section $\Sigma \subset \widetilde{U}$ of $\widetilde{U} \to A$ with $x \in \Sigma$ and a neighborhood Δ of the identity in Mp so that the map $\Delta \times \Sigma \to \widetilde{U}$, $(g,s) \mapsto sg^{-1}$, realizes $\Delta \times \Sigma$ as a neighborhood $\widetilde{V}$ of x which has empty intersection with $\widetilde{D}_{\mathrm{Sp}}$. By construction $\widetilde{V} \cap \widetilde{\mathrm{H}}(W^s)$ is connected and is itself simply connected. Thus the desired unique extension of ϕ exists on $\widetilde{V}$. At x this extension is simply defined by taking limits of ϕ along arbitrary sequences $\{x_n\}$ from $\widetilde{\mathrm{H}}(W^s)$. Thus the extended function (still called ϕ) is well-defined on the closure of $\widetilde{\mathrm{H}}(W^s)$ and is real-analytic on the complement of $\widetilde{D}_{\mathrm{Sp}}$ in every Mp-orbit in that closure. It extends as a continuous function on the full closure of $\widetilde{\mathrm{H}}(W^s)$ by defining it to be identically ∞ on $\widetilde{D}_{\mathrm{Sp}}$, i.e., its reciprocal vanishes identically at these points. ☐

Now let us proceed with our main objective of defining the semigroup representation on $\widetilde{\mathrm{H}}(W^s)$. Recall the involution $\psi : \mathrm{H}(W^s) \to \mathrm{H}(W^s)$, $h \mapsto \sigma(h)^{-1}$, and its lift $\widetilde{\psi}$ to $\widetilde{\mathrm{H}}(W^s)$. The following will be of use at several points in the sequel.

Proposition 3.10. — $\phi \circ \widetilde{\psi} = \overline{\phi}$.

Proof. — By direct calculation, $f \circ \psi = \overline{f}$. Thus, since $f = \phi^2$, we have either $\phi \circ \widetilde{\psi} = \overline{\phi}$ or $\phi \circ \widetilde{\psi} = -\overline{\phi}$. The latter is not the case, as ϕ is not purely imaginary on the nonempty set $\mathrm{Fix}(\widetilde{\psi})$. ☐

The semigroup representation $R : \widetilde{\mathrm{H}}(W^s) \to \mathrm{End}(\mathscr{A}_V)$ will be given by a certain averaging process involving the standard representation of the Heisenberg group. The latter representation is defined as follows. For elements $w = v + cv$ of the real vector space $W_{\mathbb{R}}$, the operator $\delta(v) + \mu(cv)$ is self-adjoint and its exponential

$$T_{v+cv} := \mathrm{e}^{\mathrm{i}\delta(v)+\mathrm{i}\mu(cv)}$$

converges and is unitary (see, e.g., [**9**]). These operators satisfy the relation

$$T_w T_{w'} = T_{w+w'}\, \mathrm{e}^{\frac{\mathrm{i}}{2}\omega(w,w')} \quad (w, w' \in W_{\mathbb{R}}) \,, \tag{3.2}$$

where $\omega := \mathrm{i}A|_{W_{\mathbb{R}}}$ is the induced real symplectic structure. If $T \mapsto T^\dagger$ denotes the adjoint operation in $\mathrm{End}(\mathscr{A}_V)$, it follows from $\delta(v)^\dagger = \mu(cv)$ that

$$T_w^\dagger = T_{-w} = T_w^{-1} \quad (w \in W_{\mathbb{R}}) \,.$$

Thus if $H := W_{\mathbb{R}} \times \mathrm{U}_1$ is equipped with the Heisenberg group multiplication law,

$$(w,z)\,(w',z') := (w + w', zz'\mathrm{e}^{\frac{\mathrm{i}}{2}\omega(w,w')}) \,,$$

then $(w,z) \mapsto z\,T_w$ is an irreducible unitary representation of H on $\mathscr{A}_V$. It is well known that up to equivariant isomorphisms there is only one such representation.

The oscillator representation $x \mapsto R(x)$ of $\widetilde{\mathrm{H}}(W^s)$ is now defined by

$$R(x) = \int_{W_{\mathbb{R}}} \gamma_x(w) T_w \,\mathrm{dvol}(w) \,, \quad \gamma_x(w) := \phi(x)\,\mathrm{e}^{-\frac{1}{4}\langle w, a_x w\rangle} \,.$$

Here dvol is the Euclidean volume element on $W_{\mathbb{R}}$ which we normalize so that

$$\int_{W_{\mathbb{R}}} \mathrm{e}^{-\frac{1}{4}\langle w,w\rangle}\,\mathrm{dvol}(w) = 1 \,.$$

It should be stressed that we often parameterize $W_{\mathbb{R}} \simeq V$ by the map $v \mapsto v + cv = w$.

Notice that by the positivity of $\mathfrak{Re}(a_x)$ the Gaussian function $w \mapsto \gamma_x(w)$ decreases rapidly, so that all integrals involved in the discussion above and below are easily seen to converge. In particular, since the unitary operator T_w (for $w \in W_{\mathbb{R}}$) has L^2-norm $\|T_w\| = 1$, it follows for any $x \in \widetilde{\mathrm{H}}(W^s)$ that

$$\|R(x)\| \le |\phi(x)| \int_{W_{\mathbb{R}}} \mathrm{e}^{-\frac{1}{4}\langle w, \mathfrak{Re}(a_x) w\rangle}\,\mathrm{dvol}(w) =: C(x) \,,$$

where the bound $C(x)$ by direct computation of the integral is a finite number:

$$C(x) = |\phi(x)|\,\mathrm{Det}^{-1/2}\,(\mathfrak{Re}(a_x)) = 2^{\dim_{\mathbb{C}} V} \left| \frac{\mathrm{Det}(\mathrm{Id}_W - h)}{\mathrm{Det}(s - h^\dagger s h)} \right|^{1/2} \,, \quad h = \tau_H(x) \,. \tag{3.3}$$

Thus $R(x)$ is a bounded linear operator on $\mathscr{A}_V$. In Proposition 3.17 we will establish the uniform bound $\|R(x)\| \le C(x) < 1$ for all $x \in M$. It is also clear that the operator $R(x)$ depends continuously on $x \in \widetilde{\mathrm{H}}(W^s)$.

The main point now is to prove the semigroup multiplication rule $R(xy) = R(x)R(y)$. For this we apply the Heisenberg multiplication formula (3.2) to the inner integral of

$$R(x)R(y) = \int_{W_\mathbb{R}} \left(\int_{W_\mathbb{R}} \gamma_x(w-w')\gamma_y(w')T_{w-w'}T_{w'}\,\mathrm{dvol}(w') \right) \mathrm{dvol}(w)$$

to see that $R(xy) = R(x)R(y)$ is equivalent to the multiplication rule $\gamma_{xy} = \gamma_x \sharp \gamma_y$ where the right-hand side means the twisted convolution

$$\gamma_x \sharp \gamma_y(w) := \int_{W_\mathbb{R}} \gamma_x(w-w')\gamma_y(w')\,\mathrm{e}^{\frac{\mathrm{i}}{2}\omega(w,w')}\,\mathrm{dvol}(w') \,. \tag{3.4}$$

For the proof of the formula $\gamma_x \sharp \gamma_y = \gamma_{xy}$, we will need to know that ϕ transforms as

$$\phi(xy) = \phi(x)\phi(y)\mathrm{Det}^{-\frac{1}{2}}(a_x + a_y) \,. \tag{3.5}$$

This transformation behavior, in turn, follows directly from the expression for the semigroup multiplication rule transferred to the upper half-space ζ_s^+; we record this expression in the following statement and refer to [**8**], p. 78, for the calculation.

Proposition 3.11. — *Identifying* $\mathrm{H}(W^s)$ *with* ζ_s^+ *by the modified Cayley transformation and denoting by* $(X,Y) \mapsto X \circ Y$ *the semigroup multiplication on* ζ_s^+*, one has*

$$X \circ Y + s = (Y+s)(X+Y)^{-1}(X+s) = X + s - (X-s)(X+Y)^{-1}(X+s) \,. \tag{3.6}$$

Given Proposition 3.11, to prove the transformation rule (3.5) just set $X = a_h$ and $Y = a_{h'}$ and note that, since the semigroup multiplication law for $\widetilde{\mathrm{H}}(W^s)$ by definition yields $a_h \circ a_{h'} = a_{hh'}$, the first expression in (3.6) implies

$$f(hh') = f(h)f(h')\mathrm{Det}^{-1}(a_h + a_{h'}) \,, \tag{3.7}$$

where $f(h) = \mathrm{Det}(a_h + s)$ as above. The transformation rule for ϕ follows by taking the square root of (3.7). As usual, the sign of the square root is determined by taking the positive sign at points of the lift of T_+ in $\widetilde{\mathrm{H}}(W^s)$.

Now we come to the main point.

Proposition 3.12. — *The twisted convolution for* $x, y \in \widetilde{\mathrm{H}}(W^s)$ *satisfies* $\gamma_x \sharp \gamma_y = \gamma_{xy}$.

Proof. — Observe that the phase factor for $w, w' \in W_\mathbb{R}$ can be reorganized as

$$\mathrm{e}^{\frac{\mathrm{i}}{2}\omega(w,w')} = \mathrm{e}^{-\frac{1}{2}A(w,w')} = \mathrm{e}^{\frac{1}{4}\langle w', sw\rangle - \frac{1}{4}\langle w, sw'\rangle} \,.$$

Inserting the definitions of γ_x and γ_y in $\gamma_x \sharp \gamma_y$ we then have

$$\gamma_x \sharp \gamma_y(w) = \phi(x)\phi(y)\,\mathrm{e}^{-\frac{1}{4}\langle w, a_x w\rangle} \int \mathrm{e}^{-\frac{1}{4}\langle w', (a_x+a_y)w'\rangle + \frac{1}{4}\langle w', (a_x+s)w\rangle + \frac{1}{4}\langle w, (a_x-s)w'\rangle}\,\mathrm{dvol}(w') \,.$$

Since $\mathfrak{Re}(a_x + a_y) > 0$, the integrand is a rapidly decreasing function of $w' \in W_\mathbb{R}$ and convergence of the integral over the domain $W_\mathbb{R}$ is guaranteed.

We now wish to explicitly compute the integral by completing the square and shifting variables. For this it is a useful preparation to write

$$\langle w', a_x w\rangle = A(w', s a_x w) \quad (w' \in W_\mathbb{R}) \,,$$

and similarly for the other terms. We then holomorphically extend the right-hand side to w' in W, and by making a shift of integration variables

$$w' \to w' + (a_x + a_y)^{-1}(a_x + s)w ,$$

we bring the convolution integral into the form

$$\gamma_x \sharp \gamma_y(w) = \mathrm{e}^{-\frac{1}{4}A(w,\, s(a_x \circ a_y)w)} \phi(x)\phi(y) \int_{W_{\mathbb{R}}} \mathrm{e}^{-\frac{1}{4}A(w',\,(sa_x + sa_y)w')} \,\mathrm{dvol}(w') ,$$

where $a_x \circ a_y = -s + (a_y + s)(a_x + a_y)^{-1}(a_x + s) = a_x - (a_x - s)(a_x + a_y)^{-1}(a_x + s)$ is the semigroup multiplication on ζ_s^+. Using $A(w, sw') = \langle w, w' \rangle$ for $w \in W_{\mathbb{R}}$ and the defining relation $a_x \circ a_y = a_{xy}$, we see that the first factor on the right-hand side is $\mathrm{e}^{-\frac{1}{4}\langle w,\, a_{xy} w \rangle}$. By the transformation rule (3.5) the integral is evaluated as

$$\int_{W_{\mathbb{R}}} \mathrm{e}^{-\frac{1}{4}\langle w',\,(a_x + a_y)w' \rangle} \,\mathrm{dvol}(w') = \mathrm{Det}^{-1/2}(a_x + a_y) = \frac{\phi(xy)}{\phi(x)\phi(y)} ,$$

and multiplying factors it follows that

$$\gamma_x \sharp \gamma_y(w) = \phi(xy)\, \mathrm{e}^{-\frac{1}{4}\langle w,\, a_{xy} w \rangle} = \gamma_{xy}(w) ,$$

which is the desired semigroup property. □

Corollary 3.5. — *The mapping* $R : \widetilde{\mathrm{H}}(W^s) \to \mathrm{End}(\mathscr{A}_V)$ *defined by*

$$R(x) = \int_{W_{\mathbb{R}}} \gamma_x(w) T_w \,\mathrm{dvol}(w)$$

is a representation of the semigroup $\widetilde{\mathrm{H}}(W^s)$.

We conclude this section by deriving a formula for the adjoint.

Proposition 3.13. — *The adjoint of* $R(x)$ *is computed as* $R(x)^\dagger = R(\widetilde{\psi}(x))$. *In particular,* $R(x)R(x)^\dagger = R(x\,\widetilde{\psi}(x))$.

Proof. — We recall the relations $\overline{\phi} = \phi \circ \widetilde{\psi}$ from Proposition 3.10 and $(a_h)^\dagger = a_{\psi(h)}$ from Remark 3.3. Since $\overline{\langle w, a_h w \rangle} = \langle w, (a_h)^\dagger w \rangle$, it follows that

$$\overline{\gamma_x} = \gamma_{\widetilde{\psi}(x)} .$$

The desired formula, $R(x)^\dagger = R(\widetilde{\psi}(x))$, now results from this equation and the identities $T_w^\dagger = T_{-w}$ and $\gamma_x(-w) = \gamma_x(w)$. With this in hand, the second statement $R(x)R(x)^\dagger = R(x\,\widetilde{\psi}(x))$ is a consequence of the semigroup property. □

3.3.3. Basic conjugation formula. — Here we compute the effect of conjugating (in the semigroup sense) operators of the form $q(w)$, $w \in W$, with operators $R(x)$ coming from the semigroup. This is an immediate consequence of an analogous result for the operators T_w. For this we first allow T_w to be defined for $w = v + \varphi \in W$ by

$$T_w := \mathrm{e}^{\mathrm{i}q(w)} = \mathrm{e}^{\mathrm{i}\delta(v)+\mathrm{i}\mu(\varphi)}\,.$$

These operators are no longer defined on Fock space, but are defined on $\mathscr{O}(V)$. They satisfy

$$T_w T_{\tilde{w}} = T_{w+\tilde{w}}\,\mathrm{e}^{-\frac{1}{2}A(w,\tilde{w})}\,. \tag{3.8}$$

Note that for $x \in \widetilde{\mathrm{H}}(W^s)$ and $w \in W$ the operators $R(x)T_w$ and $T_{\tau_H(x)w}R(x)$ are bounded on $\mathscr{A}_V$. Thus we interpret the following as a statement about operators on that space.

Proposition 3.14. — *For $w \in W$ and $x \in \widetilde{\mathrm{H}}(W^s)$ one has the relation*

$$R(x)T_w = T_{\tau_H(x)w}R(x)\,.$$

Proof. — For convenience of notation we write

$$R(x) = \phi(x)\int_{W_{\mathbb{R}}} \mathrm{e}^{-\frac{1}{4}A(\tilde{w},sa_x\tilde{w})}T_{\tilde{w}}\,\mathrm{dvol}(\tilde{w})\,.$$

Thus

$$R(x)T_w = \phi(x)\int_{W_{\mathbb{R}}} \mathrm{e}^{-\frac{1}{4}A(\tilde{w},sa_x\tilde{w})-\frac{1}{2}A(\tilde{w},w)}T_{\tilde{w}+w}\,\mathrm{dvol}(\tilde{w})\,.$$

Now let $h := \tau_H(x)$ and change variables by the translation $\tilde{w} \mapsto \tilde{w} - w + hw$. Using the definition $sa_x = (\mathrm{Id}_W + h)(\mathrm{Id}_W - h)^{-1}$ and the relation

$$A((\mathrm{Id}_W - h)w_1, (\mathrm{Id}_W + h)w_2) = -A((\mathrm{Id}_W + h)w_1, (\mathrm{Id}_W - h)w_2)$$

for all $w_1, w_2 \in W$, one simplifies the exponent to obtain

$$R(x)T_w = \phi(x)\int_{W_{\mathbb{R}}} \mathrm{e}^{-\frac{1}{4}A(\tilde{w},sa_x\tilde{w})-\frac{1}{2}A(hw,\tilde{w})}T_{hw+\tilde{w}}\,\mathrm{dvol}(\tilde{w})\,.$$

Reading (3.8) backwards one sees that this expression equals $T_{hw}R(x)$. □

The basic conjugation rule now follows immediately.

Proposition 3.15. — *For every $x \in \widetilde{\mathrm{H}}(W^s)$ and $w \in W$ it follows that*

$$R(x)q(w) = q(\tau_H(x)w)R(x)\,.$$

Proof. — Apply Proposition 3.14 for w replaced by tw and differentiate both sides of the resulting formula at $t = 0$. □

3.3.4. Spectral decomposition and operator bounds. — Numerous properties of R are derived from a precise description of the spectral decomposition of $R(x)$ for $x \in M$. Since every orbit of $\mathrm{Sp}_{\mathbb{R}}$ acting by conjugation on M has nonempty intersection with T_+, it is important to understand this decomposition when $x \in T_+$. For this we begin with the case where V is one-dimensional.

Proposition 3.16. — *Suppose that V is one-dimensional and that the T_+-action on $W = V \oplus V^*$ is given by $x \cdot (v + \varphi) = \lambda v + \lambda^{-1}\varphi$ where $\lambda > 1$. If f is a basis vector of V^* then the monomials $\{f^m\}_{m \in \mathbb{N} \cup \{0\}}$ form a basis of $\mathscr{A}_V$ and one has*

$$R(x)\, f^m = \lambda^{-m-1/2} f^m \, .$$

Proof. — First note that if $w = v + cv$, then

$$a_x w = s\frac{1+x}{1-x} \cdot (v + cv) = -\frac{1+\lambda}{1-\lambda}\, v + \frac{1+\lambda^{-1}}{1-\lambda^{-1}}\, cv = \frac{\lambda+1}{\lambda-1}(v+cv)\, . \tag{3.9}$$

Thus the Gaussian function $\gamma_x(w)$ in the present case is

$$\gamma_x(v+cv) = \phi(x)\, \mathrm{e}^{-\frac{1}{2}\frac{\lambda+1}{\lambda-1}|v|^2} \, .$$

To apply the operator T_{v+cv} to f^m we use the description

$$T_{v+cv} = \mathrm{e}^{\mathrm{i}\delta(v) + \mathrm{i}\mu(cv)} = \mathrm{e}^{\mathrm{i}\mu(cv)} \mathrm{e}^{-\frac{1}{2}|v|^2} \mathrm{e}^{\mathrm{i}\delta(v)} \, .$$

Decomposing T_{v+cv} in this way is not allowed on the Fock space, but is allowed if we regard T_{v+cv} as an operator on the full space $\mathscr{O}(V)$ of holomorphic functions. The calculations are now carried out on this larger space.

Recall that $\delta(v) f^m = m f(v) f^{m-1}$. From this we obtain the explicit expression

$$T_{v+cv} f^m = \mathrm{e}^{-\frac{1}{2}|v|^2} \mathrm{e}^{\mathrm{i}\mu(cv)} \sum_{l=0}^{m} \frac{\mathrm{i}^l}{l!} m(m-1)\cdots(m-l+1)\, f(v)^l f^{m-l} \, .$$

Our goal is to compute

$$I := \int_V \mathrm{e}^{-\frac{1}{2}\frac{\lambda+1}{\lambda-1}|v|^2} T_{v+cv} f^m \, \mathrm{dvol}(v) \, ,$$

where $\mathrm{dvol}(v)$ corresponds to $\mathrm{dvol}(w)$ by the isomorphism $V \simeq W_{\mathbb{R}}$. Expanding the exponential $\mathrm{e}^{\mathrm{i}\mu(cv)}$ and using $\mu(cv) f^m = |v|^2 f(v)^{-1} f^{m+1}$, the integral I is a sum of Gaussian expected values of terms of the form $|v|^{2k} f(v)^{-k} f(v)^l$. The only terms which survive are those with $k = l$. Thus

$$\begin{aligned} I &= f^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int_V \frac{|v|^{2k}}{k!}\, \mathrm{e}^{-\frac{\lambda}{\lambda-1}|v|^2} \, \mathrm{dvol}(v) \\ &= 2^{-1} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left(\frac{\lambda-1}{\lambda}\right)^{k+1} f^m = 2^{-1}(1-\lambda^{-1})\lambda^{-m} f^m \, . \end{aligned}$$

Now $\phi(x)^2 = \mathrm{Det}(a_x + s) = \mathrm{Det}(2s(\mathrm{Id}_W - \tau_H(x))^{-1}) = (-2/(1-\lambda))(2/(1-\lambda^{-1}))$, and $\phi(x) = 2\lambda^{-1/2}(1-\lambda^{-1})^{-1}$, since we are to take the positive square root at points $x \in T_+$. Hence, $R(x) f^m = \phi(x) 2^{-1}(1-\lambda^{-1})\lambda^{-m} f^m = \lambda^{-m-1/2} f^m$ as claimed. $\square$

Remark 3.5. — Note that as $x \in T_+$ goes to the unit element (or, equivalently, $\lambda \to 1$), the expression $R(x)\, f^m$ converges to f^m in the strong sense for all $m \in \mathbb{N} \cup \{0\}$.

Now let V be of arbitrary dimension and assume that $x \in T_+$ is diagonalized on $W = V \oplus V^*$ in a basis $\{e_1, \ldots, e_d, ce_1, \ldots, ce_d\}$ with eigenvalues $\lambda_1, \ldots \lambda_d, \lambda_1^{-1}, \ldots, \lambda_d^{-1}$ respectively. Since $x \in T_+$, we have $\lambda_i > 1$ for all i. For $f_i := ce_i$ and $m := (m_1, \ldots, m_d)$ we employ the standard multi-index notation $f^m := f_1^{m_1} \cdots f_d^{m_d}$ and $\lambda^m := \lambda_1^{m_1} \cdots \lambda_d^{m_d}$. In this case the multi-dimensional integrals split up into products of one-dimensional integrals. Thus, the following is an immediate consequence of the above.

Corollary 3.6. — *Let $x \in T_+$ be diagonal in a basis $\{e_i\}$ of V with eigenvalues λ_i $(i = 1, \ldots, d)$. If f^m is a monomial $f^m \equiv \prod_i (ce_i)^{m_i}$, then $R(x)\, f^m = \lambda^{-m-1/2} f^m$.*

One would expect the same result for the spectrum to hold for every conjugate gT_+g^{-1}, and this expectation is indeed borne out. However, in the approach we are going to take here, we first need the existence and basic properties of the oscillator representation of the metaplectic group. The following is a first step in this direction.

Proposition 3.17. — *The operator norm function* $\mathrm{Mp} \times T_+ \to \mathbb{R}^{>0}$, $(g,t) \mapsto \|R(gtg^{-1})\|$ *is bounded by a continuous* Mp*-independent function* $C(t) < 1$.

Proof. — For any $x \in \widetilde{\mathrm{H}}(W^s)$ we already have the bound $\|R(x)\| \le C(x)$ where $C(x)$ was computed in (3.3). That function $C(x)$ clearly is invariant under conjugation $x \mapsto gxg^{-1}$ by $g \in \mathrm{Mp}$. Evaluating it for the case of an element $x \equiv t \in T_+$ with eigenvalues λ_i one obtains

$$C(t) = 2^{\dim_{\mathbb{C}} V} \prod\nolimits_i \left(\lambda_i^{1/2} + \lambda_i^{-1/2}\right)^{-1} .$$

The inequality $C(t) < 1$ now follows from the fact that $\lambda_i > 1$ for all i. ☐

Since $R(x)^\dagger = R(\widetilde{\psi}(x))$ and $\|R(tg)\|^2 = \|R(tg)^\dagger R(tg)\| = \|R(g^{-1}t^2g)\|$, we infer the following estimates.

Corollary 3.7. — *For all $t \in T_+$ and $g \in \mathrm{Mp}$ one has $\|R(tg)\| < 1$ and $\|R(gt)\| < 1$.*

3.4. Representation of the metaplectic group. — Recall that we have realized the metaplectic group Mp in the boundary of the oscillator semigroup $\widetilde{\mathrm{H}}(W^s)$ and that $\widetilde{\mathrm{H}}(W^s)$ contains the lifted manifold T_+ in such a way that the neutral element $\mathrm{Id} \in \mathrm{Mp}$ is in its boundary. Here we show that for $x \in T_+$ and $g \in \mathrm{Mp}$ the limit $\lim_{x \to \mathrm{Id}} R(gx)$ is a well-defined unitary operator $R'(g)$ on Fock space and $R' : \mathrm{Mp} \to \mathrm{U}(\mathscr{A}_V)$ is a unitary representation. The basic properties of this *oscillator representation* are then used to derive important facts about the semigroup representation R.

Convergence will eventually be discussed in the so-called *bounded strong* topology* (see [**8**], p. 71). For the moment, however, we shall work with the slightly weaker notion of bounded strong topology where one only requires uniform boundedness and pointwise convergence of the operators themselves (with no mention made of their adjoints). Note that since $\|R(gx)\| < 1$ by Corollary 3.7, we need only prove the convergence of $R(gx)\,f$ on a dense set of functions $f \in \mathscr{A}_V$. Let us begin with $g = \mathrm{Id}$.

Lemma 3.4. — *If a sequence $x_n \in T_+$ converges to* $\mathrm{Id} \in \mathrm{Mp}$, *then the sequence $R(x_n)$ converges in the bounded strong topology to the identity operator on Fock space.*

Proof. — If f is any T_+-eigenfunction, the sequence $R(x_n)f$ converges to f by the explicit description of the spectrum given in Corollary 3.6. The statement then follows because the subspace generated by these functions is dense. □

Using this lemma along with the semigroup property, we now show that the limiting operators exist and are well-defined.

Proposition 3.18. — *If $x_n \in T_+$ converges to* $\mathrm{Id} \in \mathrm{Mp}$, *then for every* $g \in \mathrm{Mp}$ *the sequence of operators $R(gx_n)$ converges pointwise, i.e., $R(gx_n)f \to R'(g)f$ for all f in $\mathscr{A}_V$. The limiting operator $R'(g)$ is independent of the sequence $\{x_n\}$.*

Proof. — For any $m, n \in \mathbb{N}$ there exists some $t = t(m,n) \in T_+$ sufficiently near the identity so that $\tilde{x}_m = t^{-1}x_m$ and $\tilde{x}_n = t^{-1}x_n$ are still in T_+. By the semigroup property $R(gx_n) = R(gt)R(\tilde{x}_n)$ we then have

$$\|R(gx_m)f - R(gx_n)f\| \leq \|R(gt)\| \, \|R(\tilde{x}_m)f - R(\tilde{x}_n)f\| \, .$$

Letting $t = t(m,n) \to \mathrm{Id}$ it follows from Corollary 3.7 that

$$\|R(gx_m)f - R(gx_n)f\| \leq \|R(x_m)f - R(x_n)f\| \, .$$

Thus the Cauchy property of $R(x_n)f$ is passed on to $R(gx_n)f$ and therefore the sequence $R(gx_n)f$ converges in the Hilbert space $\mathscr{A}_V$. Let $\lim_{n\to\infty} R(gx_n)f =: R'(g)f$.

To show that the limit is well-defined, pick from T_+ another sequence $y_n \to \mathrm{Id}$, let $\lim_{n\to\infty} R(gy_n)f =: R''(g)f$, and notice that $\|R'(g)f - R''(g)f\|$ is no bigger than

$$\|R'(g)f - R(gx_n)f\| + \|R(gx_n)f - R(gy_n)f\| + \|R(gy_n)f - R''(g)f\| \, .$$

Using the same reasoning as above, the middle term is estimated as

$$\|R(gx_n)f - R(gy_n)f\| \leq \|R(x_n)f - R(y_n)f\| \leq \|R(x_n)f - f\| + \|R(y_n)f - f\| \, .$$

In the limit $n \to \infty$ this yields the desired result $R'(g) = R''(g)$. □

Remark 3.6. — Since $\|R(gx_n)\| < 1$ the sequence $R(gx_n)$ converges to $R'(g)$ in the bounded strong topology. Such convergence preserves the product of operators, which is to say that if $A_n \to A$ and $B_n \to B$, then $A_nB_n \to AB$. Indeed,

$$\|(A_nB_n - AB)f\| \leq \|A_n(B_n - B)f\| + \|(A_n - A)Bf\| \, ,$$

and convergence follows from $\|A_n\| < 1$ and $A_n \to A$, $B_n \to B$. Note in particular that if $R(gx_n) \to R'(g)$ and $R(g^{-1}x_n) \to R'(g^{-1})$ then $R(gx_n)R(g^{-1}x_n) \to R'(g)R'(g^{-1})$.

The bounded strong* topology also requires pointwise convergence of the sequence of adjoint operators. Therefore we must also consider sequences of the form $R(gx_n)^\dagger$. For this (see the proof of Theorem 3.1 below) we will use the following fact.

Lemma 3.5. — *Let $\{A_n\}$ and $\{B_n\}$ be sequences of bounded operators and let $C_n := A_nB_n$. If C_n and B_n converge pointwise with $B_n \to B$ and the sequence $\{A_n\}$ is uniformly bounded, then A_n converges pointwise on the image of B.*

Proof. — If $f \in \operatorname{im} B$ then $f = \lim f_n$ where $f_n = B_n h$ for some Hilbert vector h. Write

$$(A_m - A_n) f = A_m(f - f_m) + (A_m B_m - A_n B_n) h + A_n(f_n - f)$$

and use the uniform boundedness of A_n to show that $A_n f$ converges. □

Applying this with $A_n = R(gx_n g^{-1})$, $B_n = R(x_n)$ and $C_n = A_n B_n = R(gx_n)R(g^{-1}x_n)$, we have the following statement about convergence along the conjugate $gT_+ g^{-1}$.

Proposition 3.19. — *For $g \in \mathrm{Mp}$ and $\{x_n\}$ any sequence in T_+ with $x_n \to \mathrm{Id} \in \mathrm{Mp}$, it follows that $R(gx_n g^{-1})$ converges pointwise to $R'(g)R'(g^{-1})$.*

Next, if we take three sequences in T_+ and write

$$R(x_n y_n z_n) = R(x_n g^{-1}) R(g y_n g^{-1}) R(g z_n) \to \mathrm{Id}_{\mathscr{A}_V} , \tag{3.10}$$

then it follows that the sequence $R(x_n g^{-1})$ converges to an operator $B(g^{-1})$ on the image of $R'(g)R'(g^{-1})R'(g)$ with $B(g^{-1})R'(g)R'(g^{-1})R'(g) = \mathrm{Id}_{\mathscr{A}_V}$. In particular, the operator $R'(g)$ is injective for all $g \in \mathrm{Mp}$. Finally, we define y_n by $y_n^2 = x_n$ and write $R(gx_n) = R(gy_n g^{-1})R(gy_n)$. Taking the limit of both sides of this equation entails that

$$R'(g) = R'(g)R'(g^{-1})R'(g) , \tag{3.11}$$

and since $R'(g)$ is injective, this now allows us to reach the main goal of this section.

Theorem 3.1. — *For every $g \in \mathrm{Mp}$ and every sequence $\{x_n\} \subset T_+$ with $x_n \to \mathrm{Id}$ the sequence $\{R(gx_n)\}$ converges in the bounded strong* topology. The limit $R'(g)$ is independent of the sequence and defines a unitary representation $R' : \mathrm{Mp} \to \mathrm{U}(\mathscr{A}_V)$.*

Proof. — From (3.11) we have $R'(g)(\mathrm{Id}_{\mathscr{A}_V} - R'(g^{-1})R'(g)) = 0$ and, since $R'(g)$ is injective, $R'(g^{-1})R'(g) = \mathrm{Id}_{\mathscr{A}_V}$. Hence $R'(g^{-1})$ is surjective, and thus $R'(g) \in \mathrm{GL}(\mathscr{A}_V)$ by exchanging $g \leftrightarrow g^{-1}$. For the homomorphism property we write $R(g_1 x_n)R(g_2 y_n) = R(g_1 x_n g_2 y_n) = R(g_1 x_n g_1^{-1})R(g_1 g_2 y_n)$ and take limits to obtain $R'(g_1)R'(g_2) = R'(g_1 g_2)$.

Convergence in the bounded strong* topology also requires convergence of the adjoint. This property follows from $R(gx_n)^\dagger = R(\widetilde{\psi}(gx_n)) = R(x_n g^{-1})$ and the discussion after (3.10), since $R'(g)$ is now known to be an isomorphism. Unitarity of the representation is then immediate from $R(gx_n)^\dagger \to R'(g)^\dagger$ and $R(x_n g^{-1}) \to B(g^{-1}) = R'(g)^{-1}$.

Finally, we must show that $R' : \mathrm{Mp} \to \mathrm{U}(\mathscr{A}_V)$ is continuous. This amounts to showing that if $\{g_k\}$ is a sequence in Mp which converges to g, then $R'(g_k) f \to R'(g) f$ for any $f \in \mathscr{A}_V$. Hence, we let $\{x_n\}$ be a sequence in T_+ with $x_n \to \mathrm{Id}$ and choose $t = t(m,n)$ as in the proof of Proposition 3.18 so that

$$\|R(g_k x_m) - R(g_k x_n)\| \le \|R(g_k t)\| \, \|R(\tilde{x}_m) - R(\tilde{x}_n)\| ,$$

and then let $t \to \mathrm{Id}$. Using the uniform boundedness of $R(g_k t)$ as $t \to \mathrm{Id}$, this shows that the convergence $R(g_k x_n) \to R'(g_k)$ is uniform in g_k. Since we have $g_k x_n \to g x_n$ for every fixed n, the continuity of $x \mapsto R(x) f$ then implies that $R'(g_k) f \to R'(g) f$. □

Let us underline two important consequences.

Proposition 3.20. — *For $g_1, g_2 \in \mathrm{Mp}$ and $x \in \widetilde{\mathrm{H}}(W^s)$ it follows that*

$$R(g_1 x g_2) = R'(g_1) R(x) R'(g_2)\ .$$

Proof. — If y_m and z_n are sequences in T_+ which converge to Id, then, since $x \mapsto R(x) f$ is continuous for all f in Fock space, $R(g_1 y_m x g_2 z_n)$ converges pointwise to $R(g_1 x g_2)$. On the other hand, we have $R(g_1 y_m x g_2 z_n) = R(g_1 y_m) R(x) R(g_2 z_n)$ by the semigroup property, and the right-hand side converges pointwise to $R'(g_1) R(x) R'(g_2)$. ☐

We refer to $R' : \mathrm{Mp} \to \mathrm{U}(\mathscr{A}_V)$ as the *oscillator* representation of the metaplectic group. It has the following fundamental conjugation property.

Proposition 3.21. — *Let* $\mathrm{Mp} \times W \to W$, $(g, w) \mapsto \tau(g) w$ *denote the representation of* Mp *on W defined by first applying the covering map* $\mathrm{Mp} \to \mathrm{Sp}_\mathbb{R}$ *and then the standard representation of* Sp. *If we let W act on $\mathfrak{a}(V)$ by the Weyl representation q then*

$$R'(g) q(w) R'(g)^{-1} = q(\tau(g) w)\,.$$

Proof. — Since the inverse operator $R'(g)^{-1}$ is now available, this follows from the conjugation property at the semigroup level (see Proposition 3.15). ☐

Note that analogously we have the classical conjugation formula for the representation of the Heisenberg group on the Fock space $\mathscr{A}_V$, i.e.,

$$R'(g) T_w R'(g)^{-1} = T_{\tau(g) w}$$

for all $g \in \mathrm{Mp}$ and $w \in W_\mathbb{R}$ (see Proposition 3.14).

3.4.1. The trace-class property. — The concrete formula for the eigenvalues of $R(x)$ which is given in Proposition 3.6 shows that if $x \in T_+$, then $R(x)$ is of trace class. Using the conjugation property proved above, we now show that this holds for all $x \in \widetilde{\mathrm{H}}(W^s)$.

Proposition 3.22. — *For every $x \in \widetilde{\mathrm{H}}(W^s)$ the operator $R(x)$ is of trace class.*

Proof. — We must show that the operator $\sqrt{R(x) R(x)^\dagger}$ has finite trace. To verify this property observe that $R(x) R(x)^\dagger = R(y)$ with $y := x \widetilde{\psi}(x) \in M$. Since $y = g t^2 g^{-1}$ for some $t \in T_+$ and $\sqrt{R(g t^2 g^{-1})} = R'(g) R(t) R'(g)^{-1}$, the desired result follows from the explicit formula in Proposition 3.6 for the eigenvalues of t. ☐

Proposition 3.23. — *For every $x \in \widetilde{\mathrm{H}}(W^s)$ one has*

$$\mathrm{Tr}\, R(x) = 2^{-\dim_\mathbb{C} V} \phi(x)\ .$$

Proof. — Since the representation of the Heisenberg group H on $\mathscr{A}_V$ is irreducible, the space of bounded linear operators $\mathrm{End}(\mathscr{A}_V)^H$ that commute with the H-action is just $\mathbb{C}\mathrm{Id}_{\mathscr{A}_V}$. Now the dual of the space of trace-class operators is the space of bounded linear operators itself, where a bounded linear operator A is realized as a functional on the trace-class operators by $F_A(B) := \mathrm{Tr}\,(AB)$ (see [**9**]). Invariance of F_A by the H-representation is equivalent to the invariance of A as an operator. Thus, an H-invariant bounded linear functional on the space of trace-class operators is just a multiple of Tr.

We construct such a functional, Φ, by letting 1 be the vacuum in $\mathscr{A}_V$ and averaging the expectation in $T_w 1 \in \mathscr{A}_V$ over all $w \in W_{\mathbb{R}}$:

$$\Phi(L) := \int_{W_{\mathbb{R}}} \langle T_w 1, L\, T_w 1\rangle_{\mathscr{A}_V} \,\mathrm{dvol}(w)\ . \tag{3.12}$$

Indeed, the representation of H is unitary and by using $T_w^\dagger = T_w^{-1}$ along with the basic property $T_w T_{w'} = T_{w+w'}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\omega(w,w')}$, one shows that Φ is invariant, i.e., $\Phi(T_w L T_w^{-1}) = \Phi(L)$ for all $w \in W_{\mathbb{R}}$. If $L := 1\langle 1, \cdot\rangle_{\mathscr{A}_V}$ is the projector on the vacuum, then it is a simple matter to show that

$$\mathrm{Tr}_{\mathscr{A}_V}(L) = \sqrt{2}^{\dim W_{\mathbb{R}}}\Phi(L)\ . \tag{3.13}$$

Thus, this formula holds for any bounded linear operator L of trace class.

Now $\Phi(R(x))$ is computed as

$$\Phi(R(x)) = \int_{W_{\mathbb{R}}} \left\langle T_w 1, \int_{W_{\mathbb{R}}} \gamma_x(w') T_{w'}\,\mathrm{dvol}(w') T_w 1 \right\rangle_{\mathscr{A}_V} \mathrm{dvol}(w)\,.$$

Due to the fact that $\gamma_x(w')$ is rapidly decreasing, we may take the expectation inside the inner integral:

$$\Phi(R(x)) = \int_{W_{\mathbb{R}}}\int_{W_{\mathbb{R}}} \gamma_x(w')\,\langle T_w 1, T_{w'} T_w 1\rangle_{\mathscr{A}_V}\,\mathrm{dvol}(w')\,\mathrm{dvol}(w)\ .$$

Again using the basic properties of T_w, we see that

$$\langle T_w 1, T_{w'} T_w 1\rangle = \mathrm{e}^{\mathrm{i}\omega(w',w)}\langle 1, T_{w'} 1\rangle = \mathrm{e}^{\mathrm{i}\omega(w',w) - \frac{1}{4}\langle w', w'\rangle}\ .$$

Thus the inner integral over w' is a Gaussian integral, and evaluating it we obtain

$$\int_{W_{\mathbb{R}}} \gamma_x(w')\,\mathrm{e}^{-\frac{1}{4}\langle w',w'\rangle + \mathrm{i}\omega(w',w)}\,\mathrm{dvol}(w') = \phi(x)\mathrm{Det}^{-\frac{1}{2}}(a_x + \mathrm{Id}_W)\,\mathrm{e}^{-\langle sw, (a_x+\mathrm{Id}_W)^{-1} sw\rangle}\ .$$

Integrating this quantity with respect to $\mathrm{dvol}(w)$ we obtain

$$\Phi(R(x)) = 2^{-\dim W_{\mathbb{R}}}\phi(x)\ ,$$

and the desired result for the trace follows from (3.13) and $\dim W_{\mathbb{R}} = 2\dim_{\mathbb{C}} V$. $\square$

Proposition 3.24. — *For every P_1, P_2 in the Weyl algebra and every $x \in \widetilde{\mathrm{H}}(W^s)$ the operator $q(P_1)R(x)q(P_2)$ is of trace class on the Fock space $\mathscr{A}_V$. Furthermore, the function $\widetilde{\mathrm{H}}(W^s) \to \mathbb{C}$, $x \mapsto \mathrm{Tr}\, q(P_1)R(x)q(P_2)$, is holomorphic.*

Proof. — (Sketch) It is enough to show that $q(P)R(x)$ is of trace class for the case that P is a monomial operator, i.e., $q(P) = \mu(cv)^k\delta(v')^\ell$. We must then compute the trace of the square root of $C(y) = q(P)R(y)q(P)^\dagger$ for $y = x\tilde{\psi}(x) \in M$. For this, recall that the $\dagger$-operation interchanges multiplication and differentiation. Direct computation of $|\sqrt{C(y)}\, f^m|$ shows that this number is of the order of magnitude of $\lambda^{-m} m^{k+\ell}$ where λ is the torus element corresponding to $\sqrt{y}$. Thus the conjugated operator has effectively the same trace as $R(\sqrt{y})$ itself and in particular $\mathrm{Tr}\sqrt{C(y)} < \infty$.

Turning to holomorphicity, if we insert the equation $L(x) = q(P_1)R(x)q(P_2)$ in the formula (3.12) for the trace and use the definitions to compute the various integrals, then the function $F(x) = \Phi(L(x))$ is of the form

$$F(x) = \int_{W_{\mathbb{R}}} g(x,w)\,\mathrm{dvol}(w)\ ,$$

where $g(x,\cdot)$ is a rapidly decreasing function on $W_{\mathbb{R}}$ and $g(\cdot,w)$ is holomorphic on the semigroup $\widetilde{\mathrm{H}}(W^s)$. Since the $\overline{\partial}_x$–operator can be exchanged with the integral, it is immediate that F is holomorphic and therefore so is $\mathrm{Tr}\,q(P_1)R(x)q(P_2)$. □

3.5. Compatibility with Lie algebra representation. — We now show that the semigroup representation $R : \widetilde{\mathrm{H}}(W^s) \to \mathrm{End}(\mathscr{A}_V)$ is compatible with the $\mathfrak{sp}$-representation

$$\mathfrak{sp} \xrightarrow{\tau^{-1}} \mathfrak{w}(W) \xrightarrow{q} \mathfrak{gl}(\mathfrak{a}(V))\ , \quad \mathfrak{a}(V) = \mathrm{S}(V^*)\ .$$

Let $h \in \mathrm{H}(W^s)$ and $Y \in \mathfrak{sp}$. Then, since the semigroup $\mathrm{H}(W^s)$ is open in Sp, there exists some $\varepsilon > 0$ so that the curve $[-\varepsilon,\varepsilon] \ni t \mapsto \mathrm{e}^{tY}h$ lies in $\mathrm{H}(W^s)$. Fix some point $x \in \tau_H^{-1}(h)$ and let $t \mapsto \mathrm{e}^{tY}\cdot x$ denote the lifted curve in $\widetilde{\mathrm{H}}(W^s)$.

Lemma 3.6. — *For all $x \in \widetilde{\mathrm{H}}(W^s)$ and all $Y \in \mathfrak{sp}$ it follows that*

$$\frac{d}{dt}\bigg|_{t=0} R(\mathrm{e}^{tY}\cdot x) = q(\tau^{-1}(Y))R(x)\ .$$

Proof. — Recall that the operator $R(x)$ is the result of integrating the Heisenberg translations T_w against the Gaussian density $\gamma_x(w)\,\mathrm{dvol}(w)$. Thus

$$\frac{d}{dt}\bigg|_{t=0} R(\mathrm{e}^{tY}\cdot x) = \int_{W_{\mathbb{R}}} \frac{d}{dt}\bigg|_{t=0} \gamma_{\mathrm{e}^{tY}\cdot x}(w) T_w\,\mathrm{dvol}(w)\ . \tag{3.14}$$

For $w_1, w_2 \in W$ the linear transformation $Y : w \mapsto w_1 A(w_2,w) + w_2 A(w_1,w)$ is in $\mathfrak{sp}$, and $\mathfrak{sp}$ is spanned by such transformations. It is therefore sufficient to prove the statement of the lemma for Y of this form. Hence let $Y := w_1 A(w_2,\cdot) + w_2 A(w_1,\cdot)$ and observe that the corresponding element in the Weyl algebra is

$$\tau^{-1}(Y) = \tfrac{1}{2}(w_1 w_2 + w_2 w_1)\ .$$

Now, defining T_w for $w \in W$ by $T_w = \mathrm{e}^{\mathrm{i}q(w)}$ as before, we have

$$q(\tau^{-1}(Y))R(x) = -\frac{d^2}{dt_1 dt_2}\bigg|_{t_1=t_2=0} T_{t_1 w_1 + t_2 w_2} R(x)\ . \tag{3.15}$$

Therefore, for $\tilde{w} := t_1 w_1 + t_2 w_2$ consider the expression

$$T_{\tilde{w}} R(x) = \int_{W_{\mathbb{R}}} \gamma_x(w) T_{\tilde{w}} T_w\,\mathrm{dvol}(w)\ .$$

Using $T_{\tilde{w}} T_w = \mathrm{e}^{-\frac{1}{2}A(\tilde{w},w)} T_{\tilde{w}+w}$ and shifting integration variables $w \to w - \tilde{w}$ we obtain

$$T_{\tilde{w}} R(x) = \int_{W_{\mathbb{R}}} \gamma_x(w-\tilde{w})\,\mathrm{e}^{-\frac{1}{2}A(\tilde{w},w)} T_w\,\mathrm{dvol}(w)\ . \tag{3.16}$$

Comparing Eqs. (3.15,3.16) with (3.14) we see that the formula of the lemma is true if

$$\frac{d}{dt}\bigg|_{t=0} \gamma_{\mathrm{e}^{tY}\cdot x}(w) = -\frac{d^2}{dt_1 dt_2}\bigg|_{t_1=t_2=0} \gamma_x(w - t_1 w_1 - t_2 w_2)\, \mathrm{e}^{-\frac{1}{2}A(t_1 w_1 + t_2 w_2, w)} \ .$$

But checking this equation is just a simple matter of computing derivatives. Recall that $\gamma_x(w) = \phi(x)\,\mathrm{e}^{-\frac{1}{4}A(w, s a_x w)}$ and $\phi(x) = \mathrm{Det}^{1/2}(a_x + s)$. Writing $h := \tau_H(x)$ and using $\mathrm{Tr}\,Y = 0$ one computes the left-hand side to be

$$\frac{d}{dt}\bigg|_{t=0} \gamma_{\mathrm{e}^{tY}\cdot x}(w) = \gamma_x(w)\left(\tfrac{1}{4}\mathrm{Tr}\,(Y s a_x) + \tfrac{1}{2}A(h(1-h)^{-1}w, Yh(1-h)^{-1}w)\right) \ .$$

On substituting $Y = w_1 A(w_2, \cdot) + w_2 A(w_1, \cdot)$, this expression immediately agrees with the result of taking the two derivatives on the right-hand side. ☐

4. The extended character

Having prepared the algebraic foundations (§2) and the necessary representation-theoretic tools (§3), we now turn to the main part of our work, which is to prove the formula (1.1) for the K-Haar expectation value $I(t)$ of a product of ratios of characteristic polynomials. We will achieve this goal by exploiting the fact that $I(t)$ is the same as the character $\chi(t)$ of the irreducible $\mathfrak{g}$-representation on $\mathscr{A}_V^K$. The key property determining χ is a system of differential equations coming from the Casimir invariants of the Lie superalgebra $\mathfrak{g}$: acting as differential operators on a certain supermanifold $\mathscr{F}$ of Lie supergroup type, these invariants annihilate χ as a section of $\mathscr{F}$.

4.1. Generalities on Lie supergroup representations. —

4.1.1. Grassmann envelopes. — For a $\mathbb{Z}_2$-graded complex vector space $V = V_0 \oplus V_1$ let $\mathrm{End}(V) \equiv \mathfrak{gl}(V)$ be the Lie superalgebra with bracket $[X, X'] = XX' - (-1)^{|X||X'|}X'X$. Then, given any finite-dimensional complex vector space $\mathbb{C}^d$, define the *Grassmann envelope* of $\mathrm{End}(V)$ by the Grassmann algebra $\Omega = \wedge(\mathbb{C}^d) = \Omega_0 \oplus \Omega_1$ as

$$\mathrm{End}_\Omega(V) = \oplus_s (\Omega_s \otimes \mathrm{End}(V)_s) \ , \quad \Omega_s = \oplus_{k \geq 0} \wedge^{2k+s} (\mathbb{C}^d) \ .$$

The Grassmann envelope $\mathrm{End}_\Omega(V)$ is given the structure of an associative algebra by

$$(\omega \otimes X)\,(\omega' \otimes X') := (-1)^{|\omega||\omega'|} \omega\omega' \otimes XX' \ ,$$

and it also has the structure of a Lie algebra by the usual commutator:

$$[(\omega \otimes X), (\omega' \otimes X')] = \omega'\omega \otimes [X, X'] \ .$$

More generally, if $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \subset \mathrm{End}(V)$ is a complex Lie (sub)superalgebra, the Grassmann envelope $\tilde{\mathfrak{g}}(\Omega)$ of $\mathfrak{g}$ by Ω is still defined in the same way:

$$\tilde{\mathfrak{g}}(\Omega) := \oplus_s (\Omega_s \otimes \mathfrak{g}_s) \ ,$$

and it still carries the same structure of associative algebra and Lie algebra. The supertrace $\mathrm{STr} : \tilde{\mathfrak{g}}(\Omega) \to \Omega_0$ is defined by $\omega \otimes X \mapsto \omega\,\mathrm{STr}X$. Note that this satisfies $\mathrm{STr}\,[\xi, \eta] = 0$ for all $\xi, \eta \in \tilde{\mathfrak{g}}(\Omega)$.

The Grassmann envelope $\tilde{\mathfrak{g}}(\Omega)$ is $\mathbb{Z}_2$-graded by $\tilde{\mathfrak{g}}(\Omega) = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$ where $\tilde{\mathfrak{g}}_s = \Omega_s \otimes \mathfrak{g}_s$. There also exists a $\mathbb{Z}$-grading, which is induced by that of Ω; this is $\tilde{\mathfrak{g}}(\Omega) = \bigoplus_k \tilde{\mathfrak{g}}^{(k)}$ with $\tilde{\mathfrak{g}}^{(2l+s)} = \wedge^{2l+s}(\mathbb{C}^d) \otimes \mathfrak{g}_s$. Note that $[\tilde{\mathfrak{g}}^{(k)}, \tilde{\mathfrak{g}}^{(l)}] \subset \tilde{\mathfrak{g}}^{(k+l)}$. Hence, $\mathfrak{n} := \bigoplus_{k \geq 1} \tilde{\mathfrak{g}}^{(k)}$ is a nilpotent ideal. The subspace $\tilde{\mathfrak{g}}_0$ is a Lie subalgebra and we have $\mathfrak{g}_0 = \tilde{\mathfrak{g}}^{(0)} \subset \tilde{\mathfrak{g}}_0$. The following is an elementary statement found, e.g., in [**1**].

Lemma 4.1. — *Let $x \in \mathrm{End}_\Omega(V)$ and consider the decompositions $x = x_0 + x_1$ and $x = x^{(0)} + \ldots + x^{(d)}$ with respect to the $\mathbb{Z}_2$- and $\mathbb{Z}$-gradings of $\mathrm{End}_\Omega(V)$. The element x is invertible in $\mathrm{End}_\Omega(V)$ if and only if x_0 is invertible, and the latter is the case if and only if $x^{(0)} \in \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$.*

Let $G(\Omega)$ be any Lie group such that $\mathrm{Lie}(G(\Omega)) = \tilde{\mathfrak{g}}(\Omega)$. Following [**1**] we write

$$G = \exp(\mathfrak{g}_0) \subset G(\Omega)\,, \quad G(\Omega)_0 = \exp(\tilde{\mathfrak{g}}_0) \subset G(\Omega)\,,$$
$$G(\Omega)_1 = \exp(\tilde{\mathfrak{g}}_1)\,, \quad N(\Omega) = \exp(\mathfrak{n})\,, \quad N(\Omega)_0 = \exp(\mathfrak{n} \cap \tilde{\mathfrak{g}}_0)\,.$$

The objects G, $G(\Omega)_0$, $N(\Omega)$, and $N(\Omega)_0$ are subgroups of $G(\Omega)$. The group $N(\Omega)$ is a normal subgroup of $G(\Omega)$. Here is another elementary result from [**1**]:

Lemma 4.2. — *Each element $\tilde{g} \in G(\Omega)$ has a unique factorization of the form $\tilde{g} = g\theta_1 = \theta_2 g$ with $g \in G(\Omega)_0$ and $\theta_1, \theta_2 \in G(\Omega)_1$. Each element $g \in G(\Omega)_0$ can be uniquely represented in the form $g = g_0 n_1 = n_2 g_0$ with $g_0 \in G$ and $n_1, n_2 \in N(\Omega)_0$.*

4.1.2. Lie supergroups. — A complex *Lie supergroup* is a pair $(\mathfrak{g}, G)$ consisting of a complex Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and a complex Lie group G that exponentiates $\mathfrak{g}_0$, i.e., $\mathrm{Lie}(G) = \mathfrak{g}_0$. Given a Lie superalgebra $\mathfrak{g}$ there exist, in general, several choices of G. If $\mathfrak{g} \to \mathrm{End}(V)$ is a faithful finite-dimensional representation of $\mathfrak{g}$, i.e., $\mathfrak{g} \subset \mathrm{End}(V)$, then one choice is $G := \exp(\mathfrak{g}_0) \subset \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \subset \mathrm{End}(V)_0$.

A *superfunction* on a complex Lie supergroup $(\mathfrak{g}, G)$ is a section in the sheaf $\mathscr{F} \equiv \mathscr{F}(G, \mathfrak{g}_1)$ of germs of holomorphic functions on G with values in $\wedge(\mathfrak{g}_1^*)$. This sheaf $\mathscr{F}$ is a locally free sheaf of $\mathscr{O}_G$-modules where $\mathscr{O}_G$ is the sheaf of germs of holomorphic functions on G. The gradings on $\wedge(\mathfrak{g}_1^*)$ give a $\mathbb{Z}$-grading $\mathscr{F} = \bigoplus_k \mathscr{F}^{(k)}$ and a $\mathbb{Z}_2$-grading $\mathscr{F} = \mathscr{F}_0 \oplus \mathscr{F}_1$. If $f = f^{(0)} + \ldots + f^{(d_1)}$ is the decomposition of a superfunction f with respect to the former, then $\mathrm{num}(f) := f^{(0)}$ is called the *numerical part* of f.

A *superderivation* $D \in \mathrm{Der}\mathscr{F}$ is a linear sheaf morphism $D : \mathscr{F} \to \mathscr{F}$ satisfying the graded Leibniz rule $D(fh) = (Df)h + (-1)^{|D||f|} f(Dh)$ for $f, h \in \mathscr{F}$. The $\mathbb{Z}_2$-grading of $\mathscr{F}$ defines a $\mathbb{Z}_2$-grading on the vector space $\mathrm{Der}\mathscr{F}$ by $|D| = s$ if $D\mathscr{F}_t \subset \mathscr{F}_{s+t}$.

4.1.3. Representation of $\mathfrak{g}$ by superderivations. — In the theory of ordinary Lie groups one has a realization of the Lie algebra by left- or right-invariant vector fields acting as derivations of the algebra of differentiable functions on the Lie group. In the same spirit, one wants to construct a representation of the Lie superalgebra $\mathfrak{g}$ by superderivations on the sheaf of algebras $\mathscr{F}$. The basic tool for this construction is a correspondence between superfunctions $f \in \mathscr{F}$ and functions on $G(\Omega)$ with values in Ω. There exists no canonical correspondence of such kind; one possible choice is as follows.

Let $\{E_1,\ldots,E_{d_0}\}$ be a basis of $\mathfrak{g}_0$, and let $\{F_1,\ldots,F_{d_1}\}$ be a basis of $\mathfrak{g}_1$ with dual basis $\xi_1,\ldots,\xi_{d_1}$ of $\mathfrak{g}_1^*$. Then if $X\in\mathfrak{g}_0$ and $\hat{X}^R$ denotes the corresponding left-invariant vector field on functions φ on G,

$$(\hat{X}^R\varphi)(g) := \frac{d}{dt}\varphi(g\,\mathrm{e}^{tX})\big|_{t=0}\,,$$

we assign to a section $f=\sum_J f_J\,\xi_{j_1}\wedge\cdots\wedge\xi_{j_k}\in\mathscr{F}$ a function $\Phi_f : G(\Omega)\to\Omega$ by

$$\Phi_f(g\,\mathrm{e}^{\sum\alpha_i\otimes E_i}\mathrm{e}^{\sum\beta_j\otimes F_j}) = \sum\nolimits_J(\mathrm{e}^{\sum\alpha_i\hat{E}_i^R}f_J)(g)\,\beta_{j_1}\wedge\cdots\wedge\beta_{j_k}\,,$$

where $\alpha_i\in\Omega_0$ and $\beta_j\in\Omega_1$. The assignment $f\mapsto\Phi_f$ is injective if $\Omega=\wedge(\mathbb{C}^d)$ and $d\geq d_1$; and is so in particular if $\mathbb{C}^d=\mathfrak{g}_1^*$. In that case, the inverse mapping $\Phi_f\mapsto f$ is given by restriction to elements $\tilde{g}=g\,\mathrm{e}^{\sum\xi_j\otimes F_j}$ with $g\in G$.

The advantage of passing to the right-hand side of the correspondence $f\leftrightarrow\Phi_f$ is that functions on $G(\Omega)$ can be shifted by elements of $G(\Omega)$ by multiplication from the left or right. This possibility now puts us in a position to construct the desired representation of the Lie superalgebra $\mathfrak{g}$ by superderivations on the sheaf $\mathscr{F}(G,\mathfrak{g}_1)$.

Consider first the case $X\in\mathfrak{g}_0$. Making the choice $\Omega:=\wedge(\mathfrak{g}_1^*)$ and linearizing the function $\mathbb{R}\times G(\Omega)\to\Omega$, $(t,\tilde{g})\mapsto\Psi(\mathrm{e}^{-tX}\tilde{g})$ with respect to the parameter of the one-parameter group $t\mapsto\mathrm{e}^{tX}$, we obtain a first-order differential operator $\tilde{X}^L$:

$$\Psi(\mathrm{e}^{-tX}\tilde{g}) = \Psi(\tilde{g}) + (t\tilde{X}^L\Psi)(\tilde{g}) + \mathscr{O}(t^2)\,.$$

Similarly, $\tilde{X}^R$ is defined by $\Psi(\tilde{g}\,\mathrm{e}^{tX}) = \Psi(\tilde{g})+(t\tilde{X}^R\Psi)(\tilde{g})+\mathscr{O}(t^2)$. Derivations $\hat{X}^i$ on $\mathscr{F}$ are then given by $\hat{X}^i:=\Phi^{-1}\circ\tilde{X}^i\circ\Phi$, i.e., by the equation $\Phi_{\hat{X}^i f}=\tilde{X}^i\Phi_f$ ($i=L,R$).

To handle the case $X\in\mathfrak{g}_1$, we need one extra anti-commuting parameter for the purpose of differentiation. Denoting this parameter by τ we choose $\Omega:=\wedge(\mathfrak{g}_1^*\oplus\mathbb{C}\tau)$ and define $\tilde{X}^{L,R}$ by $\Psi(\mathrm{e}^{-\tau\otimes X}\tilde{g})=\Psi(\tilde{g})+(\tau\tilde{X}^L\Psi)(\tilde{g})$ and $\Psi(\tilde{g}\,\mathrm{e}^{\tau\otimes X})=\Psi(\tilde{g})+(\tau\tilde{X}^R\Psi)(\tilde{g})$. Again, the corresponding derivations on $\mathscr{F}$ are $\hat{X}^{L,R}:=\Phi^{-1}\circ\tilde{X}^{L,R}\circ\Phi$.

On basic grounds [**1**] one then has the following statement.

Lemma 4.3. — *The assignments $X\mapsto\hat{X}^L$ and $X\mapsto\hat{X}^R$ are representations of the Lie superalgebra $\mathfrak{g}$ on the sheaf $\mathscr{F}(G,\mathfrak{g}_1)$ of superfunctions. The two representations commute in the graded-commutative sense, i.e., $\hat{X}^L\hat{Y}^R=(-1)^{|X||Y|}\hat{Y}^R\hat{X}^L$.*

4.1.4. Representations and characters of Lie supergroups. — A *representation* of a Lie supergroup $(\mathfrak{g},G)$ on a $\mathbb{Z}_2$-graded vector space $V=V_0\oplus V_1$ is a homomorphism of Lie superalgebras $\rho_*:\mathfrak{g}\to\mathrm{End}(V)$ and a homomorphism of Lie groups $\rho: G\to\mathrm{GL}(V_0)\times\mathrm{GL}(V_1)\subset\mathrm{GL}(V)$ which are compatible in the sense that $(d\rho)_e=\rho_*|_{\mathfrak{g}_0}$.

Example 4.1. — Given a $\mathbb{Z}_2$-graded vector space $U=U_0\oplus U_1$, consider the Lie superalgebra $\mathfrak{gl}(U)$ and the Lie supergroup $(\mathfrak{gl}(U),\mathrm{GL}(U_0)\times\mathrm{GL}(U_1))$. For any integer m the two maps $\rho:\mathrm{GL}(U_0)\times\mathrm{GL}(U_1)\to\mathbb{C}^\times$, $(g_0,g_1)\mapsto\mathrm{Det}^m(g_0)\,\mathrm{Det}^{-m}(g_1)$, and $\rho_*:\mathfrak{gl}(U)\to\mathbb{C}$, $X\mapsto m\,\mathrm{STr}X$, define a representation of $(\mathfrak{gl}(U),\mathrm{GL}(U_0)\times\mathrm{GL}(U_1))$ on the $\mathbb{Z}_2$-graded one-dimensional vector space $V_0\oplus V_1$ with $V_0=\mathbb{C}$ and $V_1=\{0\}$.

The *character* χ associated with the representation $(\rho_*, \rho; V)$ of a Lie supergroup $(\mathfrak{g}, G)$ is the superfunction

$$\chi : G \to \wedge(\mathfrak{g}_1^*) , \quad g \mapsto \mathrm{STr}_V\, \rho(g)\, \mathrm{e}^{\sum \xi_j \rho_*(F_j)} ,$$

provided that the supertrace exists. Note that the numerical part $\mathrm{num}(\chi)$ of the character is just the supertrace of the Lie group representation $\rho : G \to \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$.

An element $f \in \mathscr{F}(G, \mathfrak{g}_1)$ is called *radial* if the associated function $\Phi_f : G(\Omega) \to \Omega$ satisfies $\Phi_f(g^{-1}hg) = \Phi_f(h)$ for all $g, h \in G(\Omega)$ and if $(\hat{X}^L + \hat{X}^R) f = 0$ for all $X \in \mathfrak{g}_1$.

Lemma 4.4. — *The character of a Lie supergroup representation is radial.*

Proof. — Let χ be the character associated with the representation $(\rho_*, \rho; V)$ of a Lie supergroup $(\mathfrak{g}, G)$. Then by the definition of the correspondence $\chi \leftrightarrow \Phi_\chi$ one has

$$\Phi_\chi(g_0\, \mathrm{e}^{\sum \alpha_i \otimes E_i} \mathrm{e}^{\sum \beta_j \otimes F_j}) = \mathrm{STr}_V\, \rho(g_0)\, \mathrm{e}^{\sum \alpha_i \rho_*(E_i)} \mathrm{e}^{\sum \beta_j \rho_*(F_j)} ,$$

where the notation above ($g_0 \in G$, $\alpha_i \in \Omega_0$, $\beta_j \in \Omega_1$) is being employed. This satisfies $\Phi_\chi(h) = \Phi_\chi(g^{-1}hg)$, since $\mathrm{STr}\,(XY) = \mathrm{STr}\,(YX)$ for $X, Y \in \mathrm{End}_\Omega(V)$ and the representations ρ_* and ρ are compatible. For $X \in \mathfrak{g}_1$ the infinitesimal (or linearized) version of the same argument gives $(\tilde{X}^L + \tilde{X}^R)\Phi_\chi = 0$ and hence $(\hat{X}^L + \hat{X}^R)\chi = 0$. □

4.2. Character of the spinor-oscillator representation. — In Chapter 3 we constructed a semigroup representation $R_0 : \widetilde{\mathrm{H}}(W_0^s) \to \mathrm{End}(\mathscr{A}_V)$ and a group representation $R' : \mathrm{Mp} \to \mathrm{U}(\mathscr{A}_V)$ which exponentiate the restriction of the spinor-oscillator representation ρ_* to $\mathfrak{sp} \subset \mathfrak{osp}_0$. These are compatible in the sense that $\widetilde{\mathrm{H}}(W_0^s) = \mathrm{Mp} \times M$, Mp acts on M by conjugation, and $R_0(g_1 x g_2) = R'(g_1) R_0(x) R'(g_2)$ for all $g_1, g_2 \in \mathrm{Mp}$ and $x \in \widetilde{\mathrm{H}}(W_0^s)$. The representation $\rho_*|_\mathfrak{o} : \mathfrak{o} \subset \mathfrak{osp}_0 \to \mathfrak{gl}(\mathscr{A}_V)$ exponentiates to a Lie group representation $R_1 : \mathrm{Spin}(W_1) \to \mathrm{GL}(\mathscr{A}_V)$. Note that R_1 and R_0 commute, as they act on the two different factors of the tensor product $\mathfrak{a}(V) = \wedge(V_1^*) \otimes \mathrm{S}(V_0^*)$.

Note also that the group $\mathbb{Z}_2$ acts on $\widetilde{\mathrm{H}}(W_0^s)$ and $\mathrm{Spin}(W_1)$ by deck transformations of the $2:1$ coverings $\widetilde{\mathrm{H}}(W_0^s) \to \mathrm{H}(W_0^s)$ and $\mathrm{Spin}(W_1) \to \mathrm{SO}(W_1)$. The non-trivial element of $\mathbb{Z}_2$ is represented by a sign change, $R_0 \to -R_0$ and $R_1 \to -R_1$.

4.2.1. Spinor-oscillator character as a radial superfunction. — Given the representations R_1 and R_0, we form the semigroup representation

$$R : \mathrm{Spin}(W_1) \times_{\mathbb{Z}_2} \widetilde{\mathrm{H}}(W_0^s) \to \mathrm{End}(\mathscr{A}_V) , \quad (g_1, g_0) \mapsto R_1(g_1) R_0(g_0) .$$

We already know the representation R to be compatible with $\rho_* : \mathfrak{osp} \to \mathfrak{gl}(\mathscr{A}_V)$. Now we define a superfunction γ on $\mathrm{Spin}(W_1) \times_{\mathbb{Z}_2} \widetilde{\mathrm{H}}(W_0^s)$ with values in $\wedge(\mathfrak{osp}_1^*)$ by

$$\gamma(g_1, g_0) = \mathrm{STr}_{\mathscr{A}_V}\, R(g_1, g_0)\, \mathrm{e}^{\sum \xi_j \rho_*(F_j)} ,$$

where $\{F_1, \ldots, F_{d_1}\}$ is a basis of $\mathfrak{osp}_1$ and $\{\xi_1, \ldots, \xi_{d_1}\}$ is its dual basis. We refer to γ as the spinor-oscillator character. By the circumstance that R and ρ_* are representations on the infinite-dimensional vector space $\mathscr{A}_V$ we are obliged to discuss the domain of

definition of γ. For this, notice that $\mathrm{Spin}(W_1)$ acts non-trivially only on the finite-dimensional (or spinor) part of $\mathscr{A}_V$. Expanding $\mathrm{e}^{\sum \xi_j \rho_*(F_j)}$ we obtain a finite sum

$$\mathrm{e}^{\sum \xi_j \rho_*(F_j)} = \sum_J \rho_*(P_J) \xi_{j_1} \cdots \xi_{j_k}$$

with $P_J \in \mathfrak{q}(W)$. (Here we recall that $\mathfrak{osp}_1$ can be viewed as part of the Clifford-Weyl algebra $\mathfrak{q}(W)$ by the isomorphism $\tau^{-1} : \mathfrak{osp} \to \mathfrak{s} \subset \mathfrak{q}(W)$.) By Proposition 3.24 the operator $R(g)\rho_*(P_J)$ is of trace class. Thus the character γ is defined on the full domain $\mathrm{Spin}(W_1) \times_{\mathbb{Z}_2} \widetilde{\mathrm{H}}(W_0^s)$. Moreover, $\gamma(g)$ depends analytically on g.

Does there exist any good sense in which to think of the spinor-oscillator character γ as a radial superfunction? There is no question that for every $X \in \mathfrak{osp}$ we do have superderivations $\hat{X}^L$ and $\hat{X}^R$ on $\mathscr{F}(\mathrm{Spin} \times \widetilde{\mathrm{H}})$, and the function γ by its definition as a character does satisfy $(\hat{X}^L + \hat{X}^R)\chi = 0$. However, our semigroup elements $g \in \mathrm{Spin}(W_1) \times_{\mathbb{Z}_2} \widetilde{\mathrm{H}}(W_0^s)$ do not possess an inverse in the spinor-oscillator representation and we therefore shouldn't expect such a relation as $\Phi_\chi(g^{-1}hg) = \Phi_\chi(h)$.

A substitute is this. Let $\mathrm{Spin}_\mathbb{R} \subset \mathrm{Spin}(W_1)$ be the compact real subgroup which is obtained by exponentiating the skew-symmetric degree-two elements of the Clifford algebra generated by a Euclidean vector space $W_{1,\mathbb{R}} \subset W_1$. Let $G_\mathbb{R} := \mathrm{Spin}_\mathbb{R} \times_{\mathbb{Z}_2} \mathrm{Mp}$. Then on the real submanifold $\mathrm{Spin}_\mathbb{R} \times M_{\mathrm{Sp}}$ (with M_{Sp} formerly denoted by M) the superfunction γ is $G_\mathbb{R}$-radial, since the representations R' of Mp and R_0 on $M_{\mathrm{Sp}} \subset \widetilde{\mathrm{H}}(W_0^s)$ are compatible. As a result, we will later be able to apply Berezin's theory of radial parts of Laplace-Casimir operators.

4.2.2. Numerical part of the character. — Here we show that the restriction of the numerical part of γ to a toral set $T_1 \times_{\mathbb{Z}_2} T_+$ in $\mathrm{Spin}(W_1) \times_{\mathbb{Z}_2} \widetilde{\mathrm{H}}(W_0^s)$ gives the autocorrelation function described in §1. That is, we show that the ratios of characteristic polynomials $\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\phi} k)$ and their averages with respect to a compact group K (with $k \in K \subset \mathrm{U}_N$ and ϕ being a parameter), can be expressed as supertraces of operators on the spinor-oscillator module $\mathfrak{a}(V)$ and the submodule of K-invariants $\mathfrak{a}(V)^K$.

We begin with a summary of the relevant facts. From Proposition 3.15 we recall the basic conjugation rule for the oscillator representation:

$$R_0(x) q(w) = q(\tau_H(x) w) R_0(x) ,$$

where $x \in \widetilde{\mathrm{H}}(W_0^s)$, $w \in W_0$, and $\tau_H(x) \in \mathrm{H}(W_0^s) \subset \mathrm{Sp}(W_0)$. On the side of the spinor representation, the corresponding conjugation formula is

$$R_1(y) q(w) = q(\tau_S(y)\, w) R_1(y) , \quad w \in W_1 , \, y \in \mathrm{Spin}(W_1) .$$

This defines the 2:1 covering homomorphism $\mathrm{Spin}(W_1) \to \mathrm{SO}(W_1)$, $y \mapsto \tau_S(y)$, which exponentiates the isomorphism of Lie algebras $\tau : \mathfrak{s} \cap \mathfrak{c}_2(W_1) \to \mathfrak{o}(W_1)$.

Recall also (from Proposition 3.23) the formula for the oscillator character:

$$\mathrm{Tr}\, R_0(x) = \phi(x) , \quad \phi(x)^2 = \mathrm{Det}^{-1}(\mathrm{Id}_{W_0} - \tau_H(x)) .$$

An analogous result is known for the case of the spinor representation; see, e.g., the textbook [**2**]. Defining the spinor character as the supertrace with respect to the canonical $\mathbb{Z}_2$-grading of the spinor module, one has

$$\mathrm{STr}\, R_1(y) = \psi(y)\ , \quad \psi(y)^2 = \mathrm{Det}(\mathrm{Id}_{W_1} - \tau_S(y))\ .$$

Thus the spinor character, just like the oscillator character, is a square root. By taking the supertrace over the total Fock representation space, we obtain the formula

$$\mathrm{STr}_{\mathscr{A}_V} R_1(y) R_0(x) = \phi(x)\psi(y) =: \sqrt{\frac{\mathrm{Det}(\mathrm{Id}_{W_1} - \tau_S(y))}{\mathrm{Det}(\mathrm{Id}_{W_0} - \tau_H(x))}}\ . \tag{4.1}$$

For $W_0 = W_1$, the case of our interest, $\mathrm{Spin}(W_0)$ intersects with $\widetilde{\mathrm{H}}(W_0^s)$ and the square root $\psi(y)$ is defined in such a way that $\psi(x) = \phi(x)^{-1}$ for $x \in \mathrm{Spin}(W_0) \cap \widetilde{\mathrm{H}}(W_0^s)$.

Let now $U = U_0 \oplus U_1$ be a $\mathbb{Z}_2$-graded vector space, and let $V = U \otimes \mathbb{C}^N$. The Lie group $\mathrm{GL}(V_1) \times \mathrm{GL}(V_0)$ acts on $W = (V_1 \oplus V_1^*) \oplus (V_0 \oplus V_0^*)$ by

$$(g_1, g_0).(v_1 \oplus \varphi_1 \oplus v_0 \oplus \varphi_0) = (g_1 v_1) \oplus (\varphi_1 \circ g_1^{-1}) \oplus (g_0 v_0) \oplus (\varphi_0 \circ g_0^{-1})\ .$$

This action serves to realize the group $G := (\mathrm{GL}(U_1) \times \mathrm{GL}(U_0)) \times_{\mathbb{C}^\times} \mathrm{GL}(\mathbb{C}^N)$ as a subgroup of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_0) \subset \mathrm{SO}(W_1) \times \mathrm{Sp}(W_0)$. For the purpose of letting G act in the spinor-oscillator module $\mathfrak{a}(V)$, let this representation be lifted to that of a double covering $\tilde{G} \hookrightarrow \mathrm{Spin}(W_1) \times_{\mathbb{Z}_2} \widetilde{\mathrm{H}}(W_0^s)$. The following statement gives the value of the spinor-oscillator character on $(t_1, t_0; g) \in G$ where $g \in \mathrm{GL}(\mathbb{C}^N)$ and $t_s = \mathrm{diag}(t_{s,1}, \ldots, t_{s,n})$ are diagonal matrices in $\mathrm{GL}(U_s)$ (for $s = 0, 1$).

Lemma 4.5. — *If* $\dim U_0 = \dim U_1 = n$ *and* $|t_{0,j}| > 1$ *for all* $j = 1, \ldots, n$, *then*

$$\mathrm{STr}_{\mathfrak{a}(V)} R(t_1, t_0; g) = \prod_{j=1}^{n} \sqrt{\frac{t_{1,j}}{t_{0,j}}}^{\,N} \frac{\mathrm{Det}(\mathrm{Id}_N - (t_{1,j}\, g)^{-1})}{\mathrm{Det}(\mathrm{Id}_N - (t_{0,j}\, g)^{-1})}\ .$$

Proof. — Since t_1 and t_0 are assumed to be of diagonal form, the statement holds true for the case of general n if it does so for the special case $n = 1$. Hence let $n = 1$.

In that case t_1 and t_0 are single numbers and $t_s g$ acts on $W_s = V_s \oplus V_s^* \simeq \mathbb{C}^N \oplus (\mathbb{C}^N)^*$ as $(t_s g).(v \oplus \varphi) = (t_s g v) \oplus \varphi \circ (t_s g)^{-1}$ for $s = 0, 1$. From equation (4.1) we then have

$$\mathrm{STr}_{\mathfrak{a}(V)} R(t_1, t_0; g) = \sqrt{\frac{\mathrm{Det}(\mathrm{Id}_N - t_1 g)\, \mathrm{Det}(\mathrm{Id}_N - (t_1 g)^{-1})}{\mathrm{Det}(\mathrm{Id}_N - t_0 g)\, \mathrm{Det}(\mathrm{Id}_N - (t_0 g)^{-1})}}\ ,$$

which turns into the stated formula on pulling out a factor of $\mathrm{Det}(-t_1 g)/\mathrm{Det}(-t_0 g)$ from under the square root. (Of course, the double covering of $\mathrm{GL}(U_1) \times \mathrm{GL}(U_0)$ is to be used in order to define this square root globally.) □

In the formula of Lemma 4.5 we now set $t_{1,j} = \mathrm{e}^{\mathrm{i}\psi_j}$ and $t_{0,j} = \mathrm{e}^{\phi_j}$. We then put $g^{-1} \equiv k \in K$ and integrate against Haar measure dk of unit mass on K. This integral and the summation that defines the supertrace can be interchanged, as $\mathrm{STr}_{\mathfrak{a}(V)} R(t_1, t_0; k^{-1})$ is a finite sum of power series and the conditions $\mathfrak{Re}\, \phi_j > 0$ ensure uniform and absolute convergence. The representation of K on $\mathfrak{a}(V)$ is induced by the representation of K

on V. Therefore, averaging over K with respect to Haar measure has the effect of projecting from $\mathfrak{a}(V)$ to the K-trivial isotypic component $\mathfrak{a}(V)^K$, and we arrive at

$$\mathrm{STr}_{\mathfrak{a}(V)^K} R(t_1,t_0;\mathrm{Id}) = \mathrm{e}^{(N/2)\sum_j(\mathrm{i}\psi_j-\phi_j)} \int\limits_K \prod_{j=1}^n \frac{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\mathrm{i}\psi_j}\,k)}{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\phi_j}\,k)}\,dk\,. \tag{4.2}$$

In the case of an even dimension N, the domain of definition of this formula is a complex torus $T := T_1 \times T_+$ where $T_1 = (\mathbb{C}^\times)^n$ and $T_+ \subset (\mathbb{C}^\times)^n$ is the open subset determined by the conditions $|t_{0,j}| = \mathrm{e}^{\mathfrak{Re}\,\phi_j} > 1$ for all j. For odd N we must continue to work with a double cover (also denoted by T) to take the square root $\mathrm{e}^{(N/2)\sum_j(\mathrm{i}\psi_j-\phi_j)}$.

Let $\mathfrak{g}$ be the Howe dual partner of $\mathrm{Lie}(K)$ in $\mathfrak{osp}(W)$. We know from Proposition 2.1 that $\mathfrak{g} = \mathfrak{osp}(U \oplus U^*)$ for $K = \mathrm{O}_N$ and $\mathfrak{g} = \mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)$ for $K = \mathrm{USp}_N$. Recall also from §2.6.1 that the $\mathfrak{g}$-representation on $\mathfrak{a}(V)^K$ is irreducible and of highest weight $\lambda_N = (N/2)\sum_j(\mathrm{i}\psi_j - \phi_j)$. Denote by Γ_λ the set of weights of this representation. Let $B_\gamma = (-1)^{|\gamma|}\dim \mathfrak{a}(V)^K{}_\gamma$ be the dimension of the weight space $\mathfrak{a}(V)^K{}_\gamma$ multiplied with the correct sign to form the supertrace.

Corollary 4.1. — *On the torus T we have*

$$\sum_{\gamma\in\Gamma_\lambda} B_\gamma \mathrm{e}^\gamma = \mathrm{e}^{\lambda_N} \int\limits_K \prod_{j=1}^n \frac{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\mathrm{i}\psi_j}\,k)}{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\phi_j}\,k)}\,dk\,.$$

Remark 4.1. — On the right-hand side we recognize the correlation function (see §1) which is the object of our study and, as we have explained, is related to the character of the irreducible $\mathfrak{g}$-representation on $\mathfrak{a}(V)^K$. The left-hand side gives this character (restricted to the toral set T) in the form of a weight expansion, some information about which has already been provided by Corollary 2.3 of §2.6.1.

4.3. Formula for the character. — Being an eigenfunction of (the radial parts of) the Laplace-Casimir operators that represent the center of the universal enveloping algebra $\mathsf{U}(\mathfrak{g})$, our character $\chi = \sum_\gamma B_\gamma \mathrm{e}^\gamma$ satisfies a certain system of differential equations. Here we first describe the origin and explicit form of these differential equations. We then prove that χ is determined uniquely by these in combination with the weight constraints for $\gamma \in \Gamma_\lambda$. Finally, we provide the explicit function with these properties.

4.3.1. Extended character. — Formula (4.2) and Corollary 4.1 express the character χ as a function on the toral set T. Our next step is to describe a supermanifold with a $\mathfrak{g}$-action where χ exists as a $G_\mathbb{R}$-radial superfunction and Laplace-Casimir operators can be applied. Here we give only a rough sketch, leaving the details to the reader.

First of all, the symmetry group $G_\mathbb{R}$ for χ has to be identified. Recall from the end of §4.2.1 that the good real group acting in the spinor-oscillator representation $\mathscr{A}_V$ is

$$\mathrm{Spin}(W_{1,\mathbb{R}}) \times_{\mathbb{Z}_2} \mathrm{Mp}(W_{0,\mathbb{R}}) =: G'\,,$$

which contains $K = \mathrm{O}_N$ and $K = \mathrm{USp}_N$ as subgroups. Since we are studying the character χ of the $\mathfrak{g}$-representation on the subspace $\mathscr{A}_V^K$ of K-invariants, we now seek the

subgroup $G_{\mathbb{R}} \subset G'$ which centralizes K; this means that we are asking the exponentiated version of a question which was answered at the Lie algebra level in §2.7. Here, restricting the group G' to the centralizer of K we find

$$G_{\mathbb{R}} = \begin{cases} \mathrm{Spin}((U_1 \oplus U_1^*)_{\mathbb{R}}) \times_{\mathbb{Z}_2} \mathrm{Mp}((U_0 \oplus U_0^*)_{\mathbb{R}}) & K = \mathrm{O}_N \ , \\ \mathrm{USp}(U_1 \oplus U_1^*) \times \mathrm{SO}^*(U_0 \oplus U_0^*) & K = \mathrm{USp}_N \ . \end{cases}$$

We observe that $G_{\mathbb{R}}$ for the case of $K = \mathrm{O}_N$ is just the lower-dimensional copy of G' which corresponds to U_s taking the role of V_s. We also see immediately that the Lie algebras $\mathrm{Lie}(G_{\mathbb{R}})$ coincide with the real forms described in Propositions 2.4 and 2.5.

The second object to construct is a real domain $M_{\mathbb{R}} \subset \mathrm{Spin}(W_1) \times_{\mathbb{Z}_2} \widetilde{\mathrm{H}}(W_0^s)$ with a $G_{\mathbb{R}}$-action by conjugation such that $M_{\mathbb{R}} = G_{\mathbb{R}}.T_{\mathbb{R}}$ where $T_{\mathbb{R}} \subset T$ is a real sub-torus. The choice we make for $T_{\mathbb{R}}$ is the one singled out by the parametrization $t_{1,j} = \mathrm{e}^{\mathrm{i}\psi_j}$ and $t_{0,j} = \mathrm{e}^{\phi_j}$ with real-valued coordinates ψ_j and ϕ_j. We also want the elements of $M_{\mathbb{R}}$ to commute with those of K as endomorphisms of W. By these requirements, the good real domain $M_{\mathbb{R}}$ to consider in the case of $K = \mathrm{O}_N$ is the lower-dimensional copy of $\mathrm{Spin}_{\mathbb{R}} \times M_{\mathrm{Sp}}$ which, again, corresponds to U_s replacing V_s. By the detailed analysis of §3 (where M_{Sp} was simply denoted by M) and the fact that the diagonal elements constitute a maximal torus in $\mathrm{Spin}_{\mathbb{R}}$, we infer the desired property $M_{\mathbb{R}} = G_{\mathbb{R}}.T_{\mathbb{R}}$.

In the case of $K = \mathrm{USp}_N$ the same requirements lead to the choice

$$M_{\mathbb{R}} = \mathrm{USp}(U_1 \oplus U_1^*) \times M_{\mathrm{SO}} \subset \mathrm{Spin}(W_1) \times_{\mathbb{Z}_2} \widetilde{\mathrm{H}}(W_0^s) \ ,$$

where the construction of M_{SO} is fully parallel to that of M_{Sp}: if SO denotes the complex orthogonal group of the vector space $U_0 \oplus U_0^*$, we introduce the semigroup $\mathrm{H} \subset \mathrm{SO}$ which is defined by the inequality $g^\dagger s g < g$ for $s = -\mathrm{Id}_{U_0} \oplus \mathrm{Id}_{U_0^*}$ and then take $M_{\mathrm{SO}} \subset \mathrm{H}$ to be the totally real submanifold $M_{\mathrm{SO}} \subset \mathrm{H}$ of pseudo-Hermitian elements $m = s m^\dagger s$. For this choice we easily check that $M_{\mathbb{R}} = G_{\mathbb{R}}.T_{\mathbb{R}}$.

Having constructed a manifold $M_{\mathbb{R}}$ with an action of $G_{\mathbb{R}}$ by conjugation, we now consider the supermanifold $\mathscr{F}(M_{\mathbb{R}}, \mathfrak{g}_1)$ which is the sheaf of algebras of analytic functions on $M_{\mathbb{R}}$ with values in $\wedge(\mathfrak{g}_1^*)$ where, once again, we remind the reader that $\mathfrak{g} = \mathfrak{osp}(U \oplus U^*)$ for $K = \mathrm{O}_N$ and $\mathfrak{g} = \mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)$ for $K = \mathrm{USp}_N$. By construction, for every point $x \in M_{\mathbb{R}}$ we have $T_x M_{\mathbb{R}} \otimes \mathbb{C} \simeq \mathfrak{g}_0$, the even part of the Lie superalgebra $\mathfrak{g}$. Hence by the basic principles reviewed at the beginning of this Chapter, the supermanifold $\mathscr{F}(M_{\mathbb{R}}, \mathfrak{g}_1)$ carries two representations of $\mathfrak{g}$ by superderivations ($X \mapsto \hat{X}^L$ and $X \mapsto \hat{X}^R$). The benefit from all this is that for every Casimir invariant $I \in \mathsf{U}(\mathfrak{g})$ we now get a Laplace-Casimir operator $D(I)$ on $\mathscr{F}$ by replacing each $X \in \mathfrak{g}$ in the polynomial expression for I by the corresponding differential operator $\hat{X}^L$ (or $\hat{X}^R$).

Utilizing the present setting the character χ, which was given in (4.2) as a function on T, will now be extended to a section of $\mathscr{F}(M_{\mathbb{R}}, \mathfrak{g}_1)$. By construction, the direct product $M_{\mathbb{R}} \times K$ is contained as a subspace in $\mathrm{Spin}(W_1) \times_{\mathbb{Z}_2} \widetilde{\mathrm{H}}(W_0^s)$, and by restriction of the representation R we get a mapping $R : M_{\mathbb{R}} \times K \to \mathrm{End}(\mathscr{A}_V)$ whose image still lies in the subspace of trace-class operators. A good definition of $\chi \in \mathscr{F}$ therefore is

$$\chi(m) := \mathrm{STr}_{\mathscr{A}_V^K} R(m; \mathrm{Id})\, \mathrm{e}^{\sum \xi_j \rho_*(F_j)} \ ,$$

where $\{F_j\}$ now is a basis of $\mathfrak{g}_1$ and $\{\xi_j\}$ is its dual basis. The symbol ρ_* here denotes the $\mathfrak{g}$-representation on $\mathscr{A}_V^K$. It is clear that by restricting the numerical part of $\chi(m)$ to the toral set $T_\mathbb{R} \subset M_\mathbb{R}$ we recover the function described in (4.2).

4.3.2. Weyl group. — Consider now the numerical part of the section $\chi \in \mathscr{F}$. Being a $G_\mathbb{R}$-radial function, $\mathrm{num}(\chi)$ is invariant under the action of the Weyl group W which normalizes $T_\mathbb{R}$ with respect to the $G_\mathbb{R}$-action on $M_\mathbb{R}$ by conjugation. Since the latter group action decomposes as a direct product of two factors, so does W.

For both cases ($K = \mathrm{O}_N$, USp_N) the second factor of the Weyl group W is just the permutation group S_n. As a matter of fact, conjugation of a diagonal element $t_0 \in M_{\mathrm{Sp}}$ or $t_0 \in M_{\mathrm{SO}}$ by $g \in \mathrm{Mp}((U_0 \oplus U_0^*)_\mathbb{R})$ or $g \in \mathrm{SO}^*(U_0 \oplus U_0^*)$ can return another diagonal element only by permutation of the eigenvalues $\mathrm{e}^{\phi_1}, \ldots, \mathrm{e}^{\phi_n}$ of t_0. (No inversion $\mathrm{e}^{\phi_j} \to \mathrm{e}^{-\phi_j}$ is possible, as this would mean transgressing the oscillator semigroup.) This factor S_n of W will play no important role in the following, as the expressions we will encounter are automatically invariant under such permutations.

The first factors of W are of greater significance. For the two cases of $K = \mathrm{O}_N$ and $K = \mathrm{USp}_N$ these are the Weyl groups $W_{\mathrm{SO}_{2n}}$ and $W_{\mathrm{Sp}_{2n}}$ respectively. An explicit description of these groups is as follows. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of U and decompose $U \oplus U^*$ into a direct sum of 2-planes,

$$U \oplus U^* = P_1 \oplus \ldots \oplus P_n \,,$$

where P_j is spanned by the vector e_j and the linear form $ce_j = \langle e_j, \cdot \rangle$ ($j = 1, \ldots, n$). In both cases at hand, i.e., for the symmetric form S as well the alternating form A, this is an orthogonal decomposition. The real torus under consideration is parameterized by $(\mathrm{e}^{\mathrm{i}\psi_1}, \ldots, \mathrm{e}^{\mathrm{i}\psi_n}) \in (\mathrm{U}_1)^n$ acting by $\mathrm{e}^{\mathrm{i}\psi_j}.(e_j) = \mathrm{e}^{\mathrm{i}\psi_j} e_j$ and $\mathrm{e}^{\mathrm{i}\psi_j}.(ce_j) = \mathrm{e}^{-\mathrm{i}\psi_j} ce_j$.

The Weyl group W_{Sp} is generated by the permutations of these planes and the involutions which are defined by conjugation by the mapping that sends $e_j \mapsto ce_j$ and $ce_j \mapsto -e_j$. The Weyl group W_{SO} is generated by the permutations together with the involutions which are induced by the mappings that simply exchange e_j with ce_j. Since we are in the special orthogonal group and the determinant for a single exchange $e_j \leftrightarrow ce_j$ is -1, the number of involutions in any word in W_{SO} has to be even.

In summary, the W-action on our standard bases of linear functions, $\{\mathrm{i}\psi_j\}$ and $\{\phi_j\}$, is given by the respective permutations together with the action of the involutions defined by sign change, $\mathrm{i}\psi_j \mapsto -\mathrm{i}\psi_j$. In the sequel, the Weyl group action will be understood to be either this standard action or alternatively, depending on the context, the corresponding action on the exponentiated functions $\{\mathrm{e}^{\mathrm{i}\psi_j}\}$ and $\{\mathrm{e}^{\phi_j}\}$.

As a final remark on the subject, let us note that the Weyl group symmetries of the function $\chi(t)$ can also be read off directly from the explicit expression (4.2). In particular, the absence of reflections $\phi_j \to -\phi_j$ is clear from the conditions $\mathfrak{Re}\, \phi_j > 0$.

4.3.3. Laplace-Casimir eigenvalues. — We are now ready to start deriving a system of differential equations for χ. Recall that every Casimir element $I \in \mathsf{U}(\mathfrak{g})$ determines an invariant differential operator $D(I)$, called a Laplace-Casimir operator, by replacing each element $X \in \mathfrak{g}$ in the polynomial expression of I by $\hat{X}^L$ or $\hat{X}^R$.

Lemma 4.6. — *Let χ be the character of an irreducible representation (ρ_*, ρ) of a Lie supergroup $(\mathfrak{g}, G)$ on a complex vector space. Then for any Casimir element $I \in \mathsf{U}(\mathfrak{g})$ the character χ is an eigenfunction of the differential operator $D(I)$.*

Proof. — We have $(D(I)\chi)(g) = \mathrm{STr}\, \rho_*(I)\rho(g)\, \mathrm{e}^{\sum_j \xi_j \rho_*(F_j)}$. Since I is a Casimir element, $\rho_*(I)$ commutes with $\rho_*(X)$ for all $X \in \mathfrak{g}$. By Schur's lemma for trace-class operators, $\rho_*(I) = \lambda(I)\mathrm{Id}$ with $\lambda(I) \in \mathbb{C}$. Hence, $D(I)\chi = \lambda(I)\chi$. □

Recall now from §2.2.2 that for every $\ell \in \mathbb{N}$ we have a Casimir element $I_\ell \in \mathsf{U}(\mathfrak{osp})$ of degree 2ℓ. Recall also that under the assumption $V_0 \simeq V_1$ we introduced $\partial, \widetilde{\partial} \in \mathfrak{osp}_1$, $\Lambda = [\partial, \widetilde{\partial}] \in \mathfrak{osp}_0$, and $F_\ell \in \mathsf{U}(\mathfrak{osp})$ such that $I_\ell = [\partial, F_\ell]$ and $[\partial, \Lambda] = 0$.

Let us insert here the following comment. While $\partial, \widetilde{\partial}, \Lambda$ were abstractly defined as $\mathfrak{osp}$-generators, they acquire a transparent meaning when represented as operators on the spinor-oscillator module $\mathfrak{a}(V)$. In fact, using formula (2.8) one finds

$$\partial = \sum\nolimits_j \varepsilon(f_{1,j})\, \delta(e_{0,j}) , \quad \widetilde{\partial} = \sum\nolimits_j \mu(f_{0,j})\, \iota(e_{1,j}) .$$

Here, for notational brevity, we make no distinction between $\mathfrak{osp}$-elements ∂, $\widetilde{\partial}$, etc. and the operators representing them on $\mathfrak{a}(V)$. It is now natural to identify the spinor-oscillator module $\mathfrak{a}(V)$ (or rather, a suitable completion thereof) with the complex of holomorphic differential forms on V_0. Writing $\varepsilon(f_{1,j}) \equiv dz_j$ and $\delta(e_{0,j}) \equiv \partial/\partial z_j$ we then see that ∂ is the holomorphic exterior derivative:

$$\partial = \sum\nolimits_j dz_j \frac{\partial}{\partial z_j} .$$

$\widetilde{\partial} = \iota(v)$ becomes the operator of contraction with the vector field $v = \sum z_j\, \partial/\partial z_j$ generating scale transformations $z_j \to \mathrm{e}^t z_j$, while $\Lambda = [\partial, \widetilde{\partial}] = \partial \circ \iota(v) + \iota(v) \circ \partial$ is interpreted as the Lie derivative with respect to that vector field.

Consider now any irreducible $\mathfrak{osp}$-representation on a $\mathbb{Z}_2$-graded vector space $\mathscr{V}$ with the property that the $\mathscr{V}$-supertrace of $\mathrm{e}^{-t\Lambda}$ $(t > 0)$ exists. Let $\lambda(I_\ell)$ be the scalar value of the Casimir invariant I_ℓ in the representation $\mathscr{V}$. Then a short computation using $I_\ell = [\partial, F_\ell]$ and $[\partial, \Lambda] = 0$ shows that $\lambda(I_\ell)$ multiplied by $\mathrm{STr}_{\mathscr{V}}\, \mathrm{e}^{-t\Lambda}$ vanishes:

$$\lambda(I_\ell)\, \mathrm{STr}_{\mathscr{V}}\, \mathrm{e}^{-t\Lambda} = \mathrm{STr}_{\mathscr{V}}\, \mathrm{e}^{-t\Lambda} I_\ell = \mathrm{STr}_{\mathscr{V}}\, \mathrm{e}^{-t\Lambda}[\partial, F_\ell] = \mathrm{STr}_{\mathscr{V}}\, [\partial, \mathrm{e}^{-t\Lambda} F_\ell] = 0 ,$$

since the supertrace of any bracket is zero. Thus we are facing a dichotomy: either we have $\mathrm{STr}_{\mathscr{V}}\, \mathrm{e}^{-t\Lambda} = 0$, or else $\lambda(I_\ell) = 0$ for all $\ell \in \mathbb{N}$. Now our representation $\mathfrak{a}(V)^K$ realizes the latter alternative, which leads to the following consequence.

Lemma 4.7. — *Assuming that $U = U_0 \oplus U_1$ is a $\mathbb{Z}_2$-graded vector space with $U_0 \simeq U_1$, let $\chi \in \mathscr{F}(M_\mathbb{R}, \mathfrak{g}_1)$ be the character of the irreducible $\mathfrak{g}$-representation on $\mathfrak{a}(V)^K$ for $V = U \otimes \mathbb{C}^N$. Then $D(I_\ell)\chi = 0$ for all $\ell \in \mathbb{N}$.*

Remark 4.2. — The condition $U_0 \simeq U_1$ is needed in order for the formula $I_\ell = [\partial, F_\ell]$ of Lemma 2.9 to be available.

Proof. — For any real parameter $t > 0$ the supertrace of the operator $\mathrm{e}^{-t\Lambda}$ on $\mathfrak{a}(V)^K$ certainly exists and is non-zero. In fact, using formula (4.2) one computes the value as

$$\mathrm{STr}_{\mathfrak{a}(V)^K}\,\mathrm{e}^{-t\Lambda} = \mathrm{STr}\,R(\mathrm{e}^t\mathrm{Id}_n\,,\mathrm{e}^t\mathrm{Id}_n\,;\mathrm{Id}_N) = \int_K \frac{\mathrm{Det}^n(\mathrm{Id}_N - \mathrm{e}^{-t}k)}{\mathrm{Det}^n(\mathrm{Id}_N - \mathrm{e}^{-t}k)}\,dk = 1 \neq 0\,.$$

The dichotomy of $\lambda(I_\ell)\,\mathrm{STr}_{\mathfrak{a}(V)^K}\,\mathrm{e}^{-t\Lambda} = 0$ therefore gives $D(I_\ell)\chi = \lambda(I_\ell)\chi = 0$. □

4.3.4. Differential equations for the character. — Since the Casimir elements commute with all elements of the Lie superalgebra, the Laplace-Casimir operators leave the set of radial superfunctions invariant. We denote by $\dot{D}(I_\ell)$ the radial parts of the Laplace-Casimir operators $D(I_\ell)$ for $\ell \in \mathbb{N}$. These operators, which arise by restricting the Laplace-Casimir operators to the space of radial functions, are given by differential operators on the torus $T_\mathbb{R}$. They have been described by Berezin [**1**], and his results will now be stated in a form well adapted to our purposes.

Recall from §2.2.1 that $\mathfrak{osp}$-roots are of the form $\pm\vartheta_{sj} \pm \vartheta_{tk}$. Recall also the relations $\vartheta_{0j} = \phi_j$ and $\vartheta_{1j} = \mathrm{i}\psi_j$ ($j = 1,\ldots,n$). In the following we regard the variables ϕ_j and ψ_j as real (local) coordinates for the real torus $T_\mathbb{R}$.

For $\ell \in \mathbb{N}$ let D_ℓ be the degree-2ℓ differential operator

$$D_\ell = \sum_{j=1}^n \frac{\partial^{2\ell}}{\partial\psi_j^{2\ell}} - (-1)^\ell \sum_{j=1}^n \frac{\partial^{2\ell}}{\partial\phi_j^{2\ell}}\,.$$

Let J be the function defined on a dense open subset of $T_\mathbb{R}$ by

$$J(t) = \frac{\prod_{\alpha\in\Delta_0^+} 2\sinh\frac{\alpha(\ln t)}{2}}{\prod_{\beta\in\Delta_1^+} 2\sinh\frac{\beta(\ln t)}{2}}\,,$$

where $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$ is a system of even and odd positive roots (cf. §2.6.2).

Proposition 4.1. — *Up to a constant multiple the radial part of the differential operator $D(I_\ell)$ is of the form $\dot{D}(I_\ell) = J^{-1}(D_\ell + Q_{\ell-1}) \circ J$ where $Q_{\ell-1}$ is some polynomial (with constant coefficients) in the operators D_k of total degree less than that of D_ℓ.*

Proof. — The first statement is a direct consequence of Theorem 3.2 of [**1**] using the explicit form of the Casimir elements I_ℓ defined in equation (2.5) and the relations $\vartheta_{1j} = \mathrm{i}\psi_j$ and $\vartheta_{0j} = \phi_j$. The statement about the lower-order terms $Q_{\ell-1}$ is from Theorem 4.5 of [**1**]. □

Corollary 4.2. — *The function $\chi : T_\mathbb{R} \to \mathbb{C}$ given by $\chi(t) = \mathrm{STr}_{\mathfrak{a}(V)^K} R(t;\mathrm{Id}_N)$ satisfies the system of differential equations $D_\ell(J\chi) = 0$ for all $\ell \in \mathbb{N}$.*

Proof. — From Lemma 4.7 we have $D(I_\ell)\chi = 0$ and hence $\dot{D}(I_\ell)\chi|_{T_\mathbb{R}} = 0$ for all $\ell \in \mathbb{N}$. It follows by induction that $D_\ell(J\chi) = 0$ for all $\ell \in \mathbb{N}$. □

The methods of this section can be used to derive differential equations for the character of a certain class of irreducible representations of $\mathfrak{gl}(U) \simeq \mathfrak{g}^{(0)}$. Define

$$J_0 = \frac{\prod_{j<k} 4\sinh\frac{\mathrm{i}(\psi_j-\psi_k)}{2}\,\sinh\frac{\phi_j-\phi_k}{2}}{\prod_{j,k} 2\sinh\frac{\phi_j-\mathrm{i}\psi_k}{2}}\,.$$

Here, $\{\mathrm{i}(\psi_j-\psi_k), \phi_j-\phi_k \mid j<k\}$ and $\{\phi_j - \mathrm{i}\psi_k\}$ are the sets of even and odd positive roots of $\mathfrak{g}^{(0)}$. The following statement is Corollary 4.12 of [**4**] adapted to the present context and notation. The idea of the proof is the same as that of Lemma 4.7.

Corollary 4.3. — *Let χ be the character of an irreducible representation of the Lie supergroup $(\mathfrak{gl}(U), \mathrm{GL}(U_0)\times\mathrm{GL}(U_1))$ on a finite-dimensional $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. If $U_0 \simeq U_1$ but $\dim(V_0) \neq \dim(V_1)$, then $D_\ell(J_0\chi) = 0$ for all $\ell \in \mathbb{N}$.*

4.3.5. Uniqueness theorem. — Recall that the main goal of this paper can be stated as that of explicitly computing the character χ of a local representation defined on the space of invariants $\mathfrak{a}(V)^K$ in the spinor-oscillator module. We have restricted ourselves to the cases where K is either O_N or USp_N. The representation on $\mathfrak{a}(V)^K$ is defined at the infinitesimal level on the full complex Lie superalgebra $\mathfrak{g}$ which is the Howe partner of K in the canonical realization of $\mathfrak{osp}$ in the Clifford-Weyl algebra of $V \oplus V^*$.

It has been shown that $\chi : T_\mathbb{R} \to \mathbb{C}$ satisfies the differential equations $D_\ell(J\chi) = 0$. By analytic continuation it satisfies the same equations on the complex torus T.

Recall that Γ_λ denotes the set of weights of the $\mathfrak{g}$-representation on $\mathfrak{a}(V)^K$. Recall also from Corollary 2.3 that the weights $\gamma = \sum_{j=1}^n (\mathrm{i}m_j\psi_j - n_j\phi_j) \in \Gamma_\lambda$ satisfy the weight constraints $-\frac{N}{2} \leq m_j \leq \frac{N}{2} \leq n_j$. The highest weight is $\lambda = \frac{N}{2}\sum(\mathrm{i}\psi_j - \phi_j)$.

By the definition of the torus T the weights $\gamma \in \Gamma_\lambda$ are analytically integrable and we now view e^γ as a function on T.

Theorem 4.1. — *The character $\chi : T \to \mathbb{C}$ is annihilated by the differential operators $D_\ell \circ J$ for all $\ell \in \mathbb{N}$, and it has a convergent expansion $\chi = \sum B_\gamma \mathrm{e}^\gamma$ where the sum runs over weights $\gamma = \sum_{j=1}^n (\mathrm{i}m_j\psi_j - n_j\phi_j)$ satisfying the constraints $-\frac{N}{2} \leq m_j \leq \frac{N}{2} \leq n_j$. For the case of $K = \mathrm{USp}_N$ it is the unique function on T with these two properties and $B_\lambda = 1$. For $K = \mathrm{O}_N$ it is the unique W-invariant function on T with these two properties and $B_\lambda = 1$, $B_{\lambda - \mathrm{i}N\psi_n} = 0$.*

Remark 4.3. — To verify the property $B_{\lambda-\mathrm{i}N\psi_n} = 0$ which holds for the case of $K = \mathrm{O}_N$, look at the right-hand side of the formula of Corollary 4.1: in order to generate a term $\mathrm{e}^\gamma = \mathrm{e}^{\lambda - \mathrm{i}N\psi_n}$ in the weight expansion, you must pick the term $\mathrm{e}^{-\mathrm{i}N\psi_n}$ in the expansion of the determinant for $j = n$ in the numerator; but the latter term depends on k as $\mathrm{Det}(-k)$ which vanishes upon taking the Haar average for $K = \mathrm{O}_N$. By W-invariance the property $B_{\lambda-\mathrm{i}N\psi_n} = 0$ is equivalent to $B_{\lambda-\mathrm{i}N\psi_j} = 0$ for all j.

In view of this Remark and Corollaries 2.3 and 4.2, it is only the uniqueness statement of Theorem 4.1 that remains to be proved here. This requires a bit of preparation, in particular to appropriately formulate the condition $D_\ell(J\chi) = 0$. For that we develop $J\chi$ in a series $J\chi = \sum_\tau a_\tau f_\tau$ where the f_τ are D_ℓ-eigenfunctions for *every* $\ell \in \mathbb{N}$.

The first step is to determine an appropriate expansion for J. Recall that

$$J = \frac{\prod_{\alpha\in\Delta_0^+}(\mathrm{e}^{\frac{\alpha}{2}} - \mathrm{e}^{-\frac{\alpha}{2}})}{\prod_{\beta\in\Delta_1^+}(\mathrm{e}^{\frac{\beta}{2}} - \mathrm{e}^{-\frac{\beta}{2}})} .$$

Given a factor in the denominator of this representation, we wish to factor out, e.g., $\mathrm{e}^{-\frac{\beta}{2}}$ to obtain a term $(1-\mathrm{e}^{-\beta})^{-1}$ which we will attempt to develop in a geometric series. In order for this to converge uniformly on compact subsets of T it is necessary and sufficient for $\mathfrak{Re}\,\beta$ to be positive on $\mathfrak{t}$. This of course depends on the root β.

Fortunately, the sets of odd positive roots for our two cases of $K = \mathrm{O}_N$ and $K = \mathrm{USp}_N$ are the same (see §2.6.2):

$$\Delta_1^+ = \{\phi_j \pm \mathrm{i}\psi_k \mid j,k = 1,\ldots,n\} .$$

So indeed, if we factor out $\mathrm{e}^{-\frac{\beta}{2}}$ from each term in the denominator and do the same in the numerator, we obtain the expression

$$J = \mathrm{e}^{\delta}\,\frac{\prod_{\alpha\in\Delta_0^+}(1-\mathrm{e}^{-\alpha})}{\prod_{\beta\in\Delta_1^+}(1-\mathrm{e}^{-\beta})} ,$$

and it is possible to expand each term of the denominator in a geometric series. Here

$$\delta = \frac{1}{2}\sum_{\alpha\in\Delta_0^+}\alpha - \frac{1}{2}\sum_{\beta\in\Delta_1^+}\beta$$

is half the *supersum* of the positive roots.

Now let $\{\sigma_1,\ldots,\sigma_r\}$ be a basis of simple positive roots (cf. §2.6.2) and expand the terms $(1-\mathrm{e}^{-\beta})^{-1}$ in geometric series to obtain

$$J = \mathrm{e}^{\delta}\sum_{b\geq 0} A_b\,\mathrm{e}^{b\sigma} ,$$

which converges uniformly on compact subsets of T. In this expression b and σ denote the vectors $b = (b_1,\ldots,b_r)$ and $\sigma = (\sigma_1,\ldots,\sigma_r)$, respectively, and $b\sigma := \sum b_i\sigma_i$. Following the usual multi-index notation, $b \geq 0$ means $b_i \geq 0$ for all i. Note $A_0 = 1$.

Now we know that the character has a convergent series representation

$$\chi = \sum_{\gamma\in\Gamma_\lambda} B_\gamma\,\mathrm{e}^{\gamma} .$$

Thus we may write

$$J\chi = \sum_{\gamma\in\Gamma_\lambda} B_\gamma \sum_{b\geq 0} A_b\,\mathrm{e}^{\delta+\gamma+b\sigma} .$$

For convenience of the discussion we let $\tilde{\gamma} := \gamma + b\sigma$ and reorganize the sums as

$$J\chi = \sum_{\tilde{\gamma}}\Big(\sum A_b\,B_{\tilde{\gamma}-b\sigma}\Big)\,\mathrm{e}^{\delta+\tilde{\gamma}} , \tag{4.3}$$

where the inner sum is a finite sum which runs over all $b \geq 0$ such that $\tilde{\gamma} - b\sigma \in \Gamma_\lambda$.

We are now in a position to explain the recursion procedure which shows that χ is unique. Start by applying D_ℓ to $J\chi$ as represented in the expression (4.3). Since $\delta+\tilde{\gamma}$ is of the form $\sum(\mathrm{i}m_k\psi_k - n_k\phi_k)$, we immediately see that it is an eigenfunction with eigenvalue $E(\ell,\tilde{\gamma}) := (-1)^\ell \sum(m_k^{2\ell} - n_k^{2\ell})$. The functions $\mathrm{e}^{\delta+\tilde{\gamma}}$ in the sum are independent eigenfunctions. Hence it follows that

$$0 = E(\ell,\tilde{\gamma}) \sum A_b B_{\tilde{\gamma}-b\sigma} \tag{4.4}$$

for all $\tilde{\gamma}$ fixed and then for all $\ell \in \mathbb{N}$.

From now on we consider the equations (4.4) only in those cases where $\tilde{\gamma}$ is itself a weight of our representation. (We have license to do so as only the uniqueness part of Theorem 4.1 remains to be proved.) In this case we have the following key fact.

Lemma 4.8. — *If $\gamma \in \Gamma_\lambda$ and the eigenvalue $E(\ell,\gamma)$ vanishes for all $\ell \in \mathbb{N}$, then γ is the highest weight λ.*

Proof. — Our first job is to compute δ. For the list of even and odd positive roots we refer the reader to §2.6.2. Direct computation shows that if $K = \mathrm{O}_N$, then

$$\delta = \sum_{k=1}^{n} (k-1)(\mathrm{i}\psi_{n-k+1} - \phi_k)\ .$$

The same computation for the case of $K = \mathrm{USp}_N$ shows that

$$\delta = \sum_{k=1}^{n} k\,(\mathrm{i}\psi_k - \phi_{n-k+1})\ .$$

Now we write $\gamma = \sum_k(\mathrm{i}m_k\psi_k - n_k\phi_k)$ with the weight constraints $-\frac{N}{2} \le m_k \le \frac{N}{2} \le n_k$. The assumption that $E(\ell,\gamma)$ vanishes for all ℓ means that

$$\sum_k (m_{n-k+1}+k-1)^{2\ell} = \sum_k (n_k+k-1)^{2\ell} \quad \text{for all } \ell$$

in the case of $K = \mathrm{O}_N$. In the case of $K = \mathrm{USp}_N$ it means that

$$\sum_k (m_{n-k+1}+k)^{2\ell} = \sum_k (n_k+k)^{2\ell} \quad \text{for all } \ell\ .$$

In the second case the only solution for m_k and n_k satisfying the weight constraints is the highest weight λ itself. In the first case there is one other solution, namely that which is obtained from the highest weight by replacing $m_n = \frac{N}{2}$ by $m_n = -\frac{N}{2}$. However, one directly checks that in the O_N case, where $2\mathrm{i}\psi_n$ is not a root, it is not possible to obtain such a γ by adding some combination of roots from $\mathfrak{g}^{(2)}$ to λ. $\square$

We are now able to give the proof of the uniqueness statement of Theorem 4.1.

Proof. — We will determine B_γ recursively, starting from $B_\lambda = 1$. Let $\gamma \neq \lambda$ be a weight that satisfies the weight constraints. Then if $K = \mathrm{USp}_N$ we know that $E(\ell,\gamma)$ is non-zero for some ℓ. It therefore follows from equation (4.4) and $A_0 = 1$ that

$$0 = B_\gamma + \sum A_b B_{\gamma-b\sigma}\ , \tag{4.5}$$

where the sum runs over all $b \neq 0$ (recall that $b \geq 0$ is always the case) such that $\gamma - b\sigma \in \Gamma_\lambda$. Since the weights $\gamma - b\sigma$ involved in the sum are smaller than γ in the natural partial order defined by the basis of simple roots, equation (4.5) defines a recursion procedure for determining all coefficients B_γ.

In the case of $K = \mathrm{O}_N$ we are confronted with the fact that the weight $\gamma = \lambda - \mathrm{i}N\psi_n$ satisfies the weight constraints and yet gives $E(\ell,\gamma) = 0$ for all ℓ. However, in this exceptional situation the conditions of Theorem 4.1 provide that $B_{\lambda-\mathrm{i}N\psi_n} = 0$. Thus the expansion coefficients B_γ are still uniquely determined by our recursion procedure. $\square$

4.3.6. Explicit solution of the differential equations. — As before let Δ^+ be a set of positive roots of $\mathfrak{g} = \mathfrak{osp}(U \oplus U^*)$ or $\mathfrak{osp}(\widetilde{U} \oplus \widetilde{U}^*)$. We now decompose these sets as

$$\Delta^+ = \Delta^+_\lambda \cup (\Delta^+ \setminus \Delta^+_\lambda)\ , \quad \Delta^+_\lambda := \{\alpha \in \Delta^+ \mid \mathfrak{g}_\alpha \subset \mathfrak{g}^{(-2)}\}\ ,$$

which means that $\Delta^+ \setminus \Delta^+_\lambda$ is a set of positive roots of $\mathfrak{g}^{(0)}$. Let Δ^+_λ be further decomposed as $\Delta^+_\lambda = \Delta^+_{\lambda,0} \cup \Delta^+_{\lambda,1}$ where $\Delta^+_{\lambda,0}$ and $\Delta^+_{\lambda,1}$ are the subsets of even and odd λ-positive roots. Then the function J has a factorization as $J = J_0\, Z^{-1} \mathrm{e}^{\delta'}$ with

$$J_0 = \frac{\prod_{\alpha \in \Delta^+_0 \setminus \Delta^+_{\lambda,0}} 2\sinh\frac{\alpha}{2}}{\prod_{\beta \in \Delta^+_1 \setminus \Delta^+_{\lambda,1}} 2\sinh\frac{\beta}{2}}\ , \quad Z = \frac{\prod_{\beta \in \Delta^+_{\lambda,1}} (1 - \mathrm{e}^{-\beta})}{\prod_{\alpha \in \Delta^+_{\lambda,0}} (1 - \mathrm{e}^{-\alpha})}\ ,$$

and $\delta' = \frac{1}{2}(\sum\alpha - \sum\beta)$ is half the supersum of λ-positive roots. For the case of $K = \mathrm{O}_N$ one finds $\delta' = -\frac{1}{2}\sum(\mathrm{i}\psi_j - \phi_j) = -\lambda_1$, while for $K = \mathrm{USp}_N$ one has $\delta' = \lambda_1$.

The Weyl group W acts on T and therefore on functions on T. Let $W_\lambda \subset W$ be the subgroup which stabilizes the highest weight $\lambda = \lambda_N$ and thus the corresponding function e^λ on T. Note that W_λ is the direct product of the permutations of the set $\{\mathrm{e}^{\phi_j}\}$ and the permutations of the set $\{\mathrm{e}^{\mathrm{i}\psi_j}\}$. The symmetrizing operator S_W from W_λ-invariant analytic functions to W-invariant analytic functions on T is given by

$$S_W(f) := \sum_{[w] \in W/W_\lambda} w(f)\ .$$

Notice that the function $\mathrm{e}^\lambda Z$ is W_λ-invariant; the symmetrized function $S_W(\mathrm{e}^\lambda Z)$ then is W-invariant. We now wish to show that this function coincides with our character χ. In this endeavor, an obstacle appears to be that $\mathrm{e}^\lambda \chi$ by Corollary 4.1 is a polynomial in the variables $\mathrm{e}^{\mathrm{i}\psi_1}, \ldots, \mathrm{e}^{\mathrm{i}\psi_n}$, whereas the function Z has poles at $\mathrm{e}^{\mathrm{i}(\psi_j + \psi_k)} = 1$. Hence our next step is to show that these poles are actually cancelled by the process of W-symmetrization.

Lemma 4.9. — *The function $S_W(\mathrm{e}^\lambda Z)$ is holomorphic on $\cap_{j=1}^n \{\mathfrak{Re}\,\phi_j > 0\}$.*

Proof. — An even root $\alpha \in \Delta^+_0$ is some linear combination of either the functions ϕ_j or the functions ψ_j. Denoting the latter subset of even roots by $\Delta^+_0(\psi) \subset \Delta^+_0$, let $\Sigma_\alpha \subset T$ for $\alpha \in \Delta^+_0(\psi)$ be the complex submanifold

$$\Sigma_\alpha := \{t \in T \mid \mathrm{e}^\alpha(t) = 1\}.$$

By definition, the function $S_W(\mathrm{e}^\lambda Z)$ is holomorphic on

$$\Big(\cap_{j=1}^n \{\mathfrak{Re}\,\phi_j > 0\}\Big) \setminus \Big(\cup_{\alpha\in\Delta_0^+(\psi)}\Sigma_\alpha\Big) .$$

Now it is a theorem of complex analysis that if a function is holomorphic outside an analytic set of complex codimension at least two, then this function is everywhere holomorphic. Therefore, since the intersection of two or more of the submanifolds Σ_α is of codimension at least two in T, it suffices to show that for any $\alpha \in \Delta_0^+(\psi)$ our function $S_W(\mathrm{e}^\lambda Z)$ extends holomorphically to

$$D_\alpha := \Sigma_\alpha \setminus \Big(\cup_{\Delta_0^+(\psi)\ni\alpha'\neq\alpha}\Sigma_{\alpha'}\Big) .$$

Hence let α be some fixed root in $\Delta_0^+(\psi)$. There exists a Weyl group element $w \in W$ and a w-invariant neighborhood U of D_α such that $w : U \to U$ is a reflection fixing the points of D_α. Let $z_\alpha : U \to \mathbb{C}$ be a complex coordinate which is transverse to D_α in the sense that $w(z_\alpha) = -z_\alpha$. Because the root α occurs at most once in the product Z, the function $S_W(\mathrm{e}^\lambda Z)$ has at most a simple pole in z_α. We may choose U in such a way that $S_W(\mathrm{e}^\lambda Z)$ is holomorphic on $U \setminus D_\alpha$. Doing so we have a unique decomposition

$$S_W(\mathrm{e}^\lambda Z) = \frac{A}{z_\alpha} + B$$

where A and B are holomorphic in U. Since $S_W(\mathrm{e}^\lambda Z)$ is W-invariant, we conclude that $w(A) = -A$ and hence $A = 0$ along D_α. ☐

Lemma 4.10. — *For all $\ell, N \in \mathbb{N}$ the function $\varphi : T \to \mathbb{C}$ defined by $\varphi = S_W(\mathrm{e}^{\lambda_N} Z)$ is a solution of the differential equation $D_\ell(J\varphi) = 0$.*

Proof. — Using $Z = \mathrm{e}^{\delta'} J_0/J$ we write $\varphi = S_W(\mathrm{e}^{\lambda_N+\delta'} J_0/J)$. Then, lifting the sum over cosets $[w] \in W/W_\lambda$ to a sum over Weyl group elements $w \in W$ we obtain

$$\mathrm{ord}(W_\lambda)\, J\varphi = J \sum_{w\in W} w(\mathrm{e}^{\lambda_N+\delta'} J_0/J) = \sum_{w\in W} \mathrm{sgn}(w)\, w(J_0\, \mathrm{e}^{\lambda_N+\delta'}) ,$$

where $w \mapsto \mathrm{sgn}(w) \in \mathbb{Z}_2 = \{\pm 1\}$ is the determinant of $w \in W \subset \mathrm{O}(\mathfrak{t}) = \mathrm{O}_{2n}$.

The factor $\mathrm{e}^{\lambda_N+\delta'}$ is the character of the representation $(\frac{N\mp 1}{2}\mathrm{STr}, \mathrm{SDet}^{\frac{N\mp 1}{2}})$ of (a double cover of) the Lie supergroup $(\mathfrak{g}^{(0)}, \mathrm{GL}(U_0)\times\mathrm{GL}(U_1))$. This representation is one-dimensional, and from Corollary 4.3 we have $D_\ell(J_0\, \mathrm{e}^{\lambda_N+\delta'}) = 0$ for all $\ell, N \in \mathbb{N}$.

The statement of the lemma now follows by applying the W-invariant differential operator D_ℓ to the formula for $\mathrm{ord}(W_\lambda) J\varphi$ above. ☐

4.3.7. Weight constraints. — Here we carry out the final step in proving the explicit formula for the character χ of our representation. Since the formula in the case of $K = \mathrm{SO}_N$ follows directly from that for $K = \mathrm{O}_N$ (see §1) and the case of $K = \mathrm{U}_N$ has been handled in [**4**], we need only discuss the cases of $K = \mathrm{O}_N$ and $K = \mathrm{USp}_N$.

In order to show that the character is indeed given by $\chi = \varphi$ with $\varphi = S_W(\mathrm{e}^{\lambda} Z)$, it remains to prove that in the series development $\varphi = \sum B_\gamma \mathrm{e}^\gamma$ of the function defined by

$$\varphi = \sum_{[w]\in W/W_\lambda} \varphi_{[w]} \ , \quad \varphi_{[w]} := \mathrm{e}^{w(\lambda_N)} \frac{\prod_{\beta\in\Delta^+_{\lambda,1}} (1-\mathrm{e}^{-w(\beta)})}{\prod_{\alpha\in\Delta^+_{\lambda,0}} (1-\mathrm{e}^{-w(\alpha)})} \ , \tag{4.6}$$

the only non-zero coefficients B_γ are those where the linear functions γ are of the form $\gamma = \sum(\mathrm{i}m_k\psi_k - n_k\phi_k)$ with $-\frac{N}{2} \le m_k \le \frac{N}{2} \le n_k$. We also have to show that $B_\gamma = 0$ in the case of the exceptional weight $\gamma = \lambda - \mathrm{i}N\psi_n$ occurring for $K = \mathrm{O}_N$.

We have shown above that $\varphi = S_W(\mathrm{e}^\lambda Z)$ is holomorphic on the product of the full complex torus of the variables $\mathrm{e}^{\mathrm{i}\psi_k}$ with the domain defined by $\mathfrak{Re}\,\phi_k > 0$. Although the individual terms $\varphi_{[w]}$ in the representation of φ have poles (which cancel in the Weyl group averaging process) we may still develop each term of φ in a series expansion; this will in fact yield the desired weight constraints.

We begin with the situation where $K = \mathrm{O}_N$. In this case $\Delta^+_{\lambda,0}$ consists of the roots $\mathrm{i}\psi_j + \mathrm{i}\psi_k$ $(j<k)$ and $\phi_j + \phi_k$ $(j \le k)$, and $\Delta^+_{\lambda,1}$ is the set of roots of the form $\mathrm{i}\psi_j + \phi_k$ $(j,k = 1,\ldots,n)$. Let us first consider the term of φ where $[w] = W_\lambda$. Its denominator can be developed in a geometric series on the region corresponding to $\mathfrak{Re}\,\phi_k > 0$ for all k. There we may write this term as

$$\varphi_{[\mathrm{Id}]} = \mathrm{e}^{\lambda_N} \prod_\beta (1-\mathrm{e}^{-\beta}) \prod_\alpha \sum_{n\ge 0} \mathrm{e}^{-n\alpha} \ .$$

Here and for the remainder of this paragraph α runs through the λ-positive even roots and β through the λ-positive odd roots.

Recall that $\lambda_N = \frac{N}{2}\sum(\mathrm{i}\psi_k - \phi_k)$, and note that all powers of $\mathrm{e}^{\mathrm{i}\psi_k}$ and e^{ϕ_k} occurring in the series expansion of

$$\prod_\beta (1-\mathrm{e}^{-\beta}) \prod_\alpha \sum_{n\ge 0} \mathrm{e}^{-n\alpha}$$

are non-positive. Thus, if $\gamma = \sum(\mathrm{i}m_k\psi_k - n_k\phi_k)$ is a weight which arises in $\varphi_{[\mathrm{Id}]}$, then $n_k \ge \frac{N}{2}$ and $m_k \le \frac{N}{2}$. In the case of the m_k this is a statement only about the term $\varphi_{[\mathrm{Id}]}$, but, since the action of the Weyl group on the variables ϕ_k is just by permutation of the indices, it follows that $n_k \ge \frac{N}{2}$ holds always, independent of the term $\varphi_{[w]}$ under consideration. Hence, we neglect the ϕ_k in our further discussion and only analyze the powers of the exponentials $\mathrm{e}^{\mathrm{i}\psi_k}$ which arise in the other terms $\varphi_{[w]}$.

Given a fixed index $k \in \{1,\ldots,n\}$ we will develop every term $\varphi_{[w]}$ on a region $R = R(k)$ defined by certain inequalities which in the case of $k=1$ are

$$\mathfrak{Re}(\mathrm{i}\psi_1) > \ldots > \mathfrak{Re}(\mathrm{i}\psi_n) > 0 \ .$$

We now discuss this case in detail.

Recall that $\mathrm{i}\psi_1$ occurs in the denominator in factors of the form

$$(1-\mathrm{e}^{-w(\alpha)})^{-1} = (1-\mathrm{e}^{-w(\mathrm{i}\psi_1)-w(\mathrm{i}\psi_j)})^{-1}$$

for $j > 1$. If $w(\mathrm{i}\psi_1) = \mathrm{i}\psi_1$, then we expand these factors just as in the case of $\varphi_{[\mathrm{Id}]}$. Convergence of the resulting series is guaranteed no matter what w does to ψ_j.

In the situation where $w(\mathrm{i}\psi_1) = -\mathrm{i}\psi_1$ we rewrite the factors in the denominator as $(1-\mathrm{e}^{-w(\alpha)})^{-1} = -\mathrm{e}^{w(\alpha)}(1-\mathrm{e}^{w(\alpha)})^{-1}$ and expand, and convergence in R is again guaranteed. Adding these series we obtain a series representation

$$\varphi = \sum_{[w]} \varphi_{[w]} = \sum_{\gamma} B_\gamma \mathrm{e}^\gamma ,$$

which is convergent on R.

Lemma 4.11. — *If $\gamma = \sum(\mathrm{i}m_k\psi_k - n_k\phi_k)$ and $B_\gamma \neq 0$, then $m_1 \leq \frac{N}{2}$.*

Proof. — If $w(\mathrm{i}\psi_1) = \mathrm{i}\psi_1$, then by the same argument as in the case of $[w] = [\mathrm{Id}]$ we see that $\mathrm{e}^{\mathrm{i}\psi_1}$ occurs in the series development of $\varphi_{[w]}$ with a power m_1 of at most $\frac{N}{2}$.

Now suppose that $w(\mathrm{i}\psi_1) = -\mathrm{i}\psi_1$. Then, following the prescription above we rewrite the ψ_1-dependent factors in $\varphi_{[w]}$ as

$$\mathrm{e}^{\frac{N}{2}w(\mathrm{i}\psi_1)} \frac{\prod_{j\geq 1}(1-\mathrm{e}^{-w(\mathrm{i}\psi_1+\phi_j)})}{\prod_{j\geq 2}(1-\mathrm{e}^{-w(\mathrm{i}\psi_1+\mathrm{i}\psi_j)})} = \mathrm{e}^{(-\frac{N}{2}+1)\mathrm{i}\psi_1} \frac{\prod_{j\geq 1}(\mathrm{e}^{-\mathrm{i}\psi_1}-\mathrm{e}^{-\phi_j})}{\prod_{j\geq 2}(\mathrm{e}^{-\mathrm{i}\psi_1}-\mathrm{e}^{-w(\mathrm{i}\psi_j)})} , \tag{4.7}$$

and expand the r.h.s. in powers of $\mathrm{e}^{-\mathrm{i}\psi_1}$. It follows that in this case $m_1 \leq -\frac{N}{2}+1$, which for any positive integer N implies that $m_1 \leq \frac{N}{2}$. □

Using Weyl group invariance, this estimate for m_1 will now yield the desired result.

Lemma 4.12. — *Suppose that $K = \mathrm{O}_N$ and let $\varphi = \sum B_\gamma \mathrm{e}^\gamma$ be the globally convergent series expansion of the proposed character $\varphi = S_W(\mathrm{e}^\lambda Z)$. Then for every weight $\gamma = \sum(\mathrm{i}m_k\psi_k - n_k\phi_k)$ with $B_\gamma \neq 0$ it follows that $-\frac{N}{2} \leq m_k \leq \frac{N}{2} \leq n_k$.*

Proof. — The inequality $n_k \geq \frac{N}{2}$ was proved above as an immediate consequence of the fact that the Weyl group W effectively acts only on the ψ_j.

Above we showed that on the region R the proposed character φ has a series development where in every γ the coefficient m_1 of $\mathrm{i}\psi_1$ is at most $\frac{N}{2}$. Recalling the fact that the function φ is holomorphic on T, we infer that $m_1 \leq \frac{N}{2}$ also holds true for the globally convergent series development $\sum B_\gamma \mathrm{e}^\gamma$.

To get the same statement for $\mathrm{i}\psi_k$ with $k \neq 1$ we just change the definition of R to $R(k)$ defined by the inequalities $\mathfrak{Re}(\mathrm{i}\psi_k) > \mathfrak{Re}(\mathrm{i}\psi_1) > \ldots > 0$. Arguing for general k as we did for $k=1$ in the above Lemma, we show that the coefficient m_k of $\mathrm{i}\psi_k$ in every γ in the series expansion of every $\varphi_{[w]}$ on $R(k)$ is at most $\frac{N}{2}$. By the holomorphic property, the same is true for the global series expansion of the proposed character φ.

Hence, to complete the proof we need only show the inequality $m_k \geq -\frac{N}{2}$. But for this it suffices to note that for every k there is an element w of the Weyl group with $w(\mathrm{i}\psi_k) = -\mathrm{i}\psi_k$. Indeed, using the Weyl invariance of φ, if there was some γ where $m_k < -\frac{N}{2}$, then the coefficient of $\mathrm{i}\psi_k$ in $w(\gamma)$ would be larger than $\frac{N}{2}$. □

To complete our work, we must prove Lemma 4.12 for the case $K = \mathrm{USp}_N$. For this we use the same notation as above for the basic linear functions, namely $\mathrm{i}\psi_k$ and ϕ_k. Here, compared to the O_N case, there are only slight differences in the λ-positive roots and the Weyl group. The only difference in the roots is in $\Delta^+_{\lambda,0}$ where $\mathrm{i}\psi_j + \mathrm{i}\psi_k$ occurs

in the larger range $j \leq k$ and $\phi_j + \phi_k$ in the smaller range $j < k$. The Weyl group acts by permutation of indices on both the $\mathrm{i}\psi_j$ and ϕ_j and by sign reversal on the $\mathrm{i}\psi_j$. In this case, as opposed to the case above where only an even number of sign reversals were allowed, every sign reversal transformation is in the Weyl group.

In order to prove Lemma 4.12 in this case, we need only go through the argument in the O_N case and make minor adjustments. In fact, the main step is to prove Lemma 4.11 and, there, the only change is that the range of j for the factor $1 - \mathrm{e}^{-\mathrm{i}(\psi_1+\psi_j)}$ is larger. This is only relevant in the case $w(\mathrm{i}\psi_1) = -\mathrm{i}\psi_1$, where we rewrite the additional denominator term $(1 - \mathrm{e}^{-w(2\mathrm{i}\psi_1)})^{-1}$ as $-\mathrm{e}^{-2\mathrm{i}\psi_1}(1 - \mathrm{e}^{-2\mathrm{i}\psi_1})^{-1}$. Hence the factor in front of the ratio of products on the r.h.s. of equation (4.7) gets an additional factor of $\mathrm{e}^{-2\mathrm{i}\psi_1}$ and now is $\mathrm{e}^{-\mathrm{i}(\frac{N}{2}+1)\psi_1}$. Thus $m_1 \leq -\frac{N}{2} - 1$ which certainly implies $m_1 \leq \frac{N}{2}$.

Let us summarize this discussion.

Theorem 4.2. — *For both $K = \mathrm{O}_N$ and $K = \mathrm{USp}_N$ every weight $\gamma = \sum(\mathrm{i}m_k\psi_k - n_k\phi_k)$ occurring in the series expansion $S_W(\mathrm{e}^\lambda Z) = \sum B_\gamma \mathrm{e}^\gamma$ obeys the weight constraints*

$$-\frac{N}{2} \leq m_k \leq \frac{N}{2} \leq n_k \quad (k = 1, \ldots, n) .$$

Moreover, using the fact that the Weyl group transformations for $K = \mathrm{O}_N$ always involve an even number of sign changes, one sees that $B_{\lambda - \mathrm{i}N\psi_n} = 0$ in that case. As a consequence of the uniqueness theorem (Theorem 4.1) we therefore have

$$\chi = S_W(\mathrm{e}^\lambda Z) = \sum_{[w] \in W/W_\lambda} \mathrm{e}^{w(\lambda_N)} \frac{\prod_{\beta \in \Delta^+_{\lambda,1}} (1 - \mathrm{e}^{-w(\beta)})}{\prod_{\alpha \in \Delta^+_{\lambda,0}} (1 - \mathrm{e}^{-w(\alpha)})}$$

in both the O_N and USp_N cases. Since the SO_N case has been handled as a consequence of the result for O_N, our work is now complete.

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